

# Comparative Dynamics

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# Comparative Dynamics

- ▶ We use phase diagrams to uncover the dynamic response to shocks.
- ▶ We study tax changes in a growth model.

# Model

The **household** solves

$$\max \int_0^{\infty} e^{-\rho t} u(c_t) dt \quad (1)$$

subject to

$$\dot{k}_t = r_t k_t + w_t - c_t - \tau_t \quad (2)$$

and  $k_0$  given.

**Firms** produce output using  $F(K, L)$ .

The **government** uses the tax revenue to finance government spending:  $G_t = \tau_t$ .

## Competitive Equilibrium

A competitive equilibrium consists of functions  $c(t), k(t), \tau(t), w(t), r(t)$  that satisfy:

1. Household: Budget constraint and

$$g(c) = \frac{r - \rho}{\sigma} \quad (3)$$

2. Firms:

$$r = f'(k) - \delta \quad (4)$$

$$w = f(k) - f'(k)k \quad (5)$$

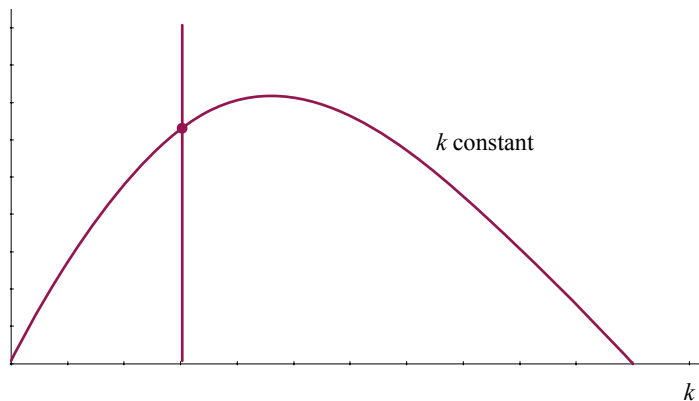
3. Government:

$$\tau = G \quad (6)$$

4. Market clearing:

$$\dot{k} = f(k) - \delta k - c - G \quad (7)$$

## Phase Diagram



The only change relative to the model without government:

## Permanent Tax Increase

Consider a permanent, unannounced increase in  $G$ .

In the phase diagram

- ▶  $\dot{k} = 0$  locus shifts down by  $\Delta G$ .
- ▶  $k_{ss}$  remains unchanged because the  $\dot{c} = 0$  locus does not shift.

Dynamics:  $c_{ss}$  drops to the new saddle path, then moves along it.

- ▶ How do I know this is true?

An interesting long-run result: full crowding out of consumption ( $\Delta c_{ss} = -\Delta G$ ).

## Temporary Tax Increase

- ▶ Consider a *temporary*, unannounced increase in  $G$ .
  - ▶  $G_t = G^* + \Delta G$  for  $0 \leq t \leq T$ , but  $G_t = G^*$  for  $t > T$ .
- ▶ To find the dynamics, we work backwards.
- ▶ Changes occur at  $t = 0$  and  $t = T$ .

Step 1:  $t = T$ . What happens?

# Temporary Tax Increase

Step 2:  $0 < t < T$ :

- ▶ The phase diagram with taxes applies.
- ▶ But the economy is not on the saddle path (why not?).
- ▶ What is the right terminal condition for  $k(T)$ ?



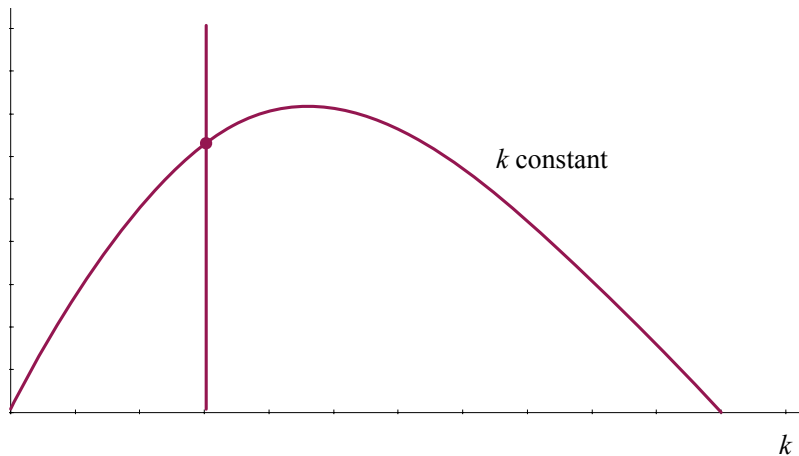
## Temporary Tax Increase

Step 3:  $t = 0$

- ▶ The  $\dot{k} = 0$  locus shifts down.
- ▶ Is  $c_0$  on the saddle path?

Consider  $k_0 = k_{ss}$ . What paths are feasible?

## Temporary Tax Increase



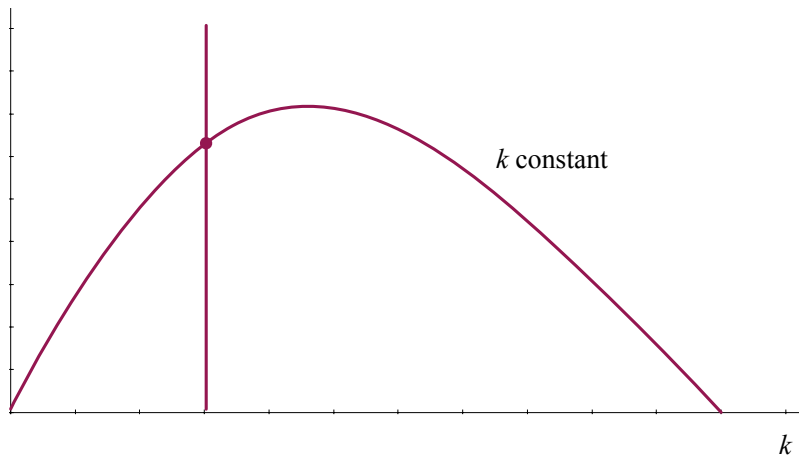
## Temporary Tax Increase

Consider  $k_0 < k_{ss}$ .

## Announced Tax Cut

- ▶ Consider a surprise tax cut that is announced to take place at date  $T$ .
- ▶ At  $t = 0$  the news arrives that taxes remain high until  $t = T$ , but then fall permanently.
- ▶ Again, we work backwards.
- ▶ Changes occur at  $t = T$  and  $t = 0$ .

## Announced Tax Cut



# Summary

To study the dynamic effects of shocks:

1. Find the phase diagram with and without shock.
2. Find the dates at which changes occur:
  - 2.1 when the shock hits: phase diagram changes  
the control (typically) does not jump
  - 2.2 when new info arrives: agents reoptimize  
the control jumps
3. Work backwards, starting at the last date at which a change occurs

# Phase Diagram for a Simple Human Capital Model

# A Human Capital Model

We study the decision of a household how much human capital to accumulate.

This example illustrates two complications:

1. finite horizons
2. binding inequality constraints.



## Household problem

The household maximizes

$$\int_0^T e^{-\rho t} u(c(t)) dt \quad (8)$$

subject to the budget constraint

$$c(t) = w(t)h(t)[1 - \tau(t)v(t)] \quad (9)$$

the human capital technology

$$\dot{h}(t) = v(t) - \delta h(t) \quad (10)$$

and  $v \geq 0$ .

For simplicity, assume that  $v \leq 1$  never binds.

## Household: Intuition

- ▶ Human capital acquired early is more valuable for two reasons:
  1. it lives longer (date  $T$  is farther off);
  2. its payoffs are discounted by less.
- ▶ We expect the optimal path for  $v(t)$  to be falling over time.
- ▶ When close to  $T$ , we expect  $v(t) \geq 0$  to bind.

# Hamiltonian

$$H = u(wh[1 - \tau v]) + \lambda [v - \delta h] \quad (11)$$

First-order conditions

$$u'(c)wh\tau \geq \lambda \quad (12)$$

with equality if  $v > 0$  and

$$\dot{\lambda} = \rho\lambda - u'(c)w(1 - \tau v) + \lambda\delta \quad (13)$$

## Summary

The solution to the household problem consists of functions  $(c, h, v, \lambda)$  that solve

1. The first-order conditions

$$u'(c)wh\tau \geq \lambda \quad (14)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - u'(c)w(1 - \tau v) \quad (15)$$

with equality if  $v > 0$ .

2. The budget constraint

$$c(t) = w(t)h(t)[1 - \tau(t)v(t)] \quad (16)$$

3. The law of motion

$$\dot{h}(t) = v(t) - \delta h(t) \quad (17)$$

4. The boundary conditions:  $h_0$  given and  $\lambda_T = 0$ .

## Log utility

Assume  $u(c) = \ln(c)$

Consider two regions of the phase diagram:

1.  $v = 0$
2.  $v > 0$

## Region $v = 0$

$$wh\tau \geq \lambda c \quad (18)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - w/c \quad (19)$$

$$c = wh \quad (20)$$

$$\dot{h} = -\delta h \quad (21)$$

Simplify:

$$\tau \geq \lambda \quad (22)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h \quad (23)$$

$$\dot{h} = -\delta h \quad (24)$$

Terminal condition:  $\lambda_T = 0$

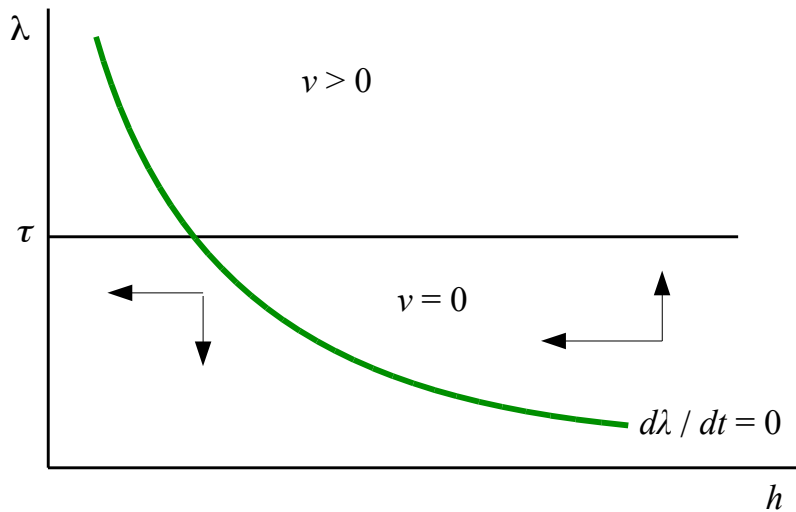
## Region $v = 0$

- ▶ The shadow price  $\lambda$  is not large enough to cover the opportunity cost  $\tau$ .
- ▶ The household does not invest in human capital.
- ▶ The laws of motion are:

$$\begin{aligned}\dot{\lambda} &= (\rho + \delta)\lambda - 1/h \\ \dot{h} &= -\delta h\end{aligned}$$

- ▶  $\lambda \uparrow \Rightarrow \dot{\lambda} \uparrow$ .
- ▶  $h \uparrow \Rightarrow \dot{\lambda} \uparrow$  and  $\dot{h} \downarrow$ .
- ▶ Hence,  $h(t) = h(t_0)e^{-\delta(t-t_0)}$ , where  $t_0$  is any date at which the economy is inside the region.

## Phase Diagram: Region $\nu = 0$





Region  $v > 0$

$$\frac{wh\tau}{c} = \lambda \quad (25)$$

$$\dot{\lambda} = (\rho + \delta)\lambda - \frac{w(1 - \tau v)}{c} \quad (26)$$

$$c = wh(1 - \tau v) \quad (27)$$

$$\dot{h} = v - \delta h \quad (28)$$

Simplify:

$$\lambda = \frac{\tau}{1 - \tau v}$$

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h$$

$$\dot{h} = v - \delta h$$

## Region $\nu > 0$

The first-order condition for  $\nu$  holds with equality:

$$\lambda(1 - \tau\nu) = \tau$$

or

$$\nu = 1/\tau - 1/\lambda \quad (29)$$

Substitute  $\nu$  out of the law of motion:

$$\dot{h} = 1/\tau - 1/\lambda - \delta h \quad (30)$$

Keep

$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h \quad (31)$$

## Region $v > 0$

- ▶ In this region, the shadow price of human capital ( $\lambda$ ) equals the opportunity cost.

$$\lambda > \tau$$

- ▶  $\dot{h} = 1/\tau - 1/\lambda - \delta h = 0$

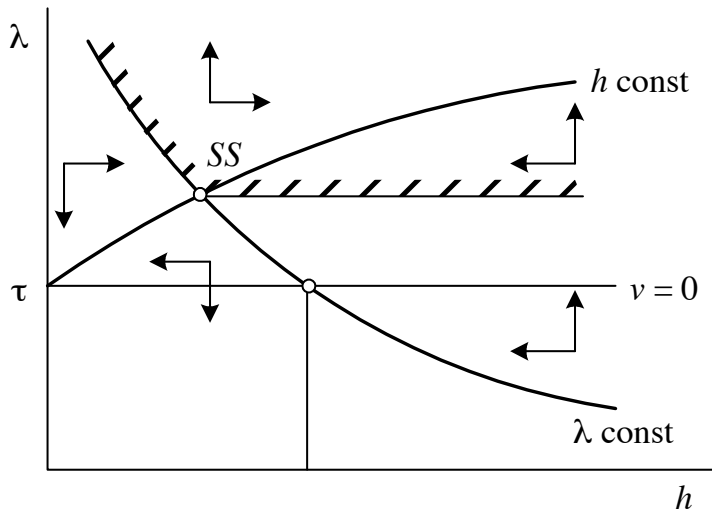
is upward sloping and starts at  $\lambda = \tau$ .

- ▶  $\dot{\lambda} = (\rho + \delta)\lambda - 1/h = 0$

is a downward sloping hyperbola (as in region  $v = 0$ ).

- ▶  $h \uparrow$  or  $\lambda \downarrow \Rightarrow \dot{h} \downarrow$ .

# Phase Diagram



## Steady State

- ▶ Assume that  $w$  and  $\tau$  are constant over time and that  $T = \infty$ .
- ▶ Then  $h$  and  $v$  converge to stationary levels,  $h_{ss}$  and  $v_{ss}$ .
- ▶ We next determine those levels.
- ▶  $\dot{\lambda} = 0$  implies

$$(\rho + \delta)h\lambda = (\rho + \delta)h\frac{\tau}{1 - \tau v} = 1 \quad (32)$$

- ▶  $\dot{h} = 0$  implies

$$v = \delta h \quad (33)$$

- ▶ Combine both

$$h_{ss} = [\tau(\rho + 2\delta)]^{-1} \quad (34)$$

## Steady State

It follows that

$$v_{ss} = \delta h_{ss} = \frac{\delta}{\tau[\rho + 2\delta]}$$

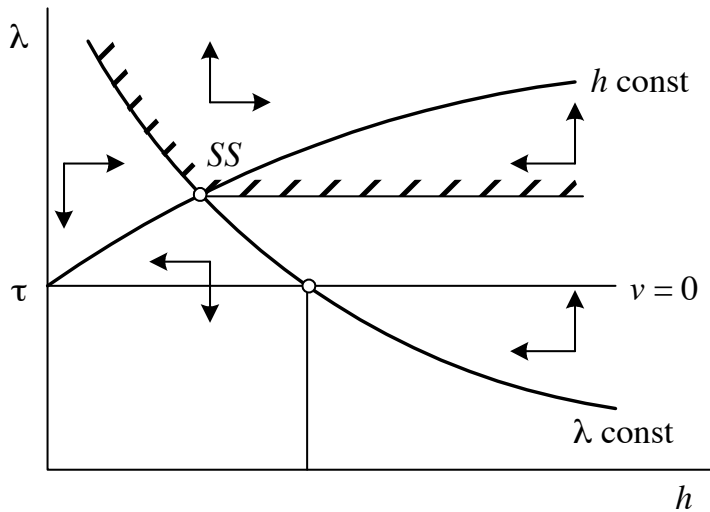
$$c_{ss} = \frac{(\rho + \delta)w}{(\rho + 2\delta)^2 \tau}$$

$$\lambda_{ss} = u' \left( \frac{(\rho + \delta)w}{(\rho + 2\delta)^2 \tau} \right) \frac{w}{\rho + 2\delta}$$

# Phase Diagram

- ▶ The phase diagram has two regions:  $v = 0$  and  $v > 0$ .
- ▶ The region boundary occurs when the household just hits the constraint  $v \geq 0$ : at  $\lambda = \tau$ .
- ▶ For  $\lambda > \tau$ :  $v > 0$ .
- ▶ For  $\lambda \leq \tau$ :  $v = 0$ .

# Phase Diagram





# Dynamics

- ▶ Any path must end with  $\lambda_T = 0$  exactly at date  $T$ .
- ▶ It follows that the shaded region must never be entered.
- ▶ What happens as the steady state is approached with  $v > 0$ ?
  - ▶ Since all the laws of motion are continuous,  $\dot{h} \rightarrow 0$  and  $\dot{\lambda} \rightarrow 0$ .
  - ▶ The steady state can never be reached.
  - ▶ But the economy can spend an arbitrarily long time arbitrarily close to the steady state.

# Dynamics

- ▶ First consider  $h_0 < h_{ss}$ .
- ▶  $\lambda$  depends on the horizon  $T$ .
- ▶ Short  $T$ :  $\lambda$  is low.
  - ▶ Start in region  $v = 0$
  - ▶ move south-west until  $\lambda_T = 0$ .

## Dynamics

- ▶ To prove this, solve the two differential equations.
- ▶  $h(t) = h(t_0) e^{-\delta t}$ . Substitute this into the law of motion for  $\lambda$  to obtain

$$\dot{\lambda} = (\rho + \delta) \lambda - e^{\delta t} / h(t_0) \quad (35)$$

- ▶ The solution to this differential equation is

$$\lambda(t) = e^{(\rho+\delta)(t-t_0)} \left[ \lambda(t_0) - \frac{\rho}{h(t_0)} \left\{ 1 - e^{-\rho(t-t_0)} \right\} \right]$$

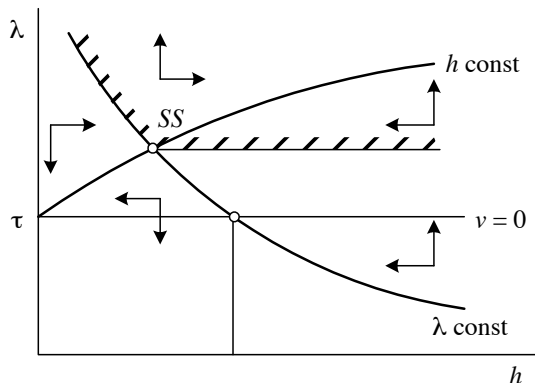
- ▶ Imposing the boundary condition  $\lambda(T) = 0$  implies  $\lambda(t_0) h(t_0) = \rho \{ 1 - e^{-\rho(T-t_0)} \}$ .
- ▶ For long  $T$ :  $\lambda(t_0) \rightarrow \rho / h(t_0)$  (unless the region  $v = 0$  is left).
- ▶ But for a short  $T$ ,  $\lambda(t_0) \rightarrow 0$ .

# Dynamics

- ▶ Case:  $h_0 < h_{ss}$  and long  $T$ .
- ▶ Initially  $v > 0$  and the economy moves south until it crosses into the  $v = 0$  region.
- ▶ As  $T \rightarrow \infty$  something bizarre happens:
  - ▶ the economy approaches the steady state without ever reaching it.
  - ▶ It comes arbitrarily close and stays arbitrarily close for an arbitrarily long time.
  - ▶ But when the terminal date comes sufficiently close it leaves the steady state and moves south-west to reach  $\lambda_T = 0$ .

# Dynamics

- ▶ Case  $h_0 > h_{SS}$ .
- ▶ Investment is never large enough to increase  $h$ .
- ▶ The economy may move straight south-west if  $T$  is short or it may move towards the steady state, similar to the case where  $h_0 < h_{SS}$ .



## Reading

- ▶ Acemoglu, Introduction to modern economic growth, ch. 8.7.
- ▶ Hendricks, Lutz (2004). "Taxation and Human Capital Accumulation." *Macroeconomic Dynamics* 8(3): 310-334.
- ▶ Sheshinski, Eytan (1968), "On the Individual's Lifetime Allocation Between Education and Work," *Metroeconomica*, 20(1), 42-9.