

# Perpetual Youth Model

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## Perpetual youth

- ▶ The standard growth model is very tractable.
- ▶ But it has an important limitation: all households are identical.
- ▶ For some questions, it is important to have households of **different ages**:
  - ▶ fiscal policies that redistribute across ages
  - ▶ models with life-cycle features: job search, matching, ...
- ▶ An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

# Demographics

- ▶ At  $t = 0$ , there are  $L(0) = 1$  identical persons.
  - ▶ They are all newborns.
- ▶ At each instant,  $nL(t)$  identical persons are born.
- ▶ Each person dies at each instant with **Poisson** probability  $\nu$ .
- ▶ The population growth rate is  $n - \nu > 0$ :

$$L(t) = \exp([n - \nu]t) \quad (1)$$

# Poisson Process

- ▶ The Poisson process is the continuous time analog of i.i.d.
- ▶ It is a counting process: it describes the distribution of the number of events occurring during a particular time interval.
- ▶ It is a one-parameter distribution, characterized by the arrival rate  $\nu$ .
- ▶ The probability of **no event** over a period of length  $\tau$  is  $\exp(-\nu\tau)$ .
- ▶ At each instant, fraction  $\nu$  of the population experiences an event.

# Demographics

- ▶ The mass of persons at  $t$  aged  $t - \tau$  is

$$\begin{aligned}L(t|\tau) &= \exp(-v(t-\tau)) \times n \exp((n-v)\tau) \\ &= \Pr(\text{live beyond } t-\tau) nL(\tau)\end{aligned}$$

- ▶ Notation:  $x(t|\tau)$  means  $x$  at  $t$  for those born at  $\tau$ .

# Preferences

- ▶ Households are indexed by  $i$ .
- ▶ Conditional on surviving, households utility at date  $t$  is  $e^{-\rho t} \ln(c_i(t))$ .
- ▶ The probability of being alive after  $t$  "periods" is  $\exp(-vt)$ .
- ▶ Expected utility for date  $t$  is  $e^{-vt} e^{-\rho t} \ln(c_i(t))$ .
- ▶ Expected lifetime utility is

$$\int_0^{\infty} e^{-(\rho+v)t} \ln(c_i(t)) dt \quad (2)$$

- ▶ Interesting: mortality simply increases the discount factor:  $\rho + v$ .

# Technology

- ▶ The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- ▶ In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \quad (3)$$

- ▶  $k = K/L$  is capital per capita and capital per worker.

# Markets

Competitive markets for

- ▶ goods (numeraire)
- ▶ labor rental:  $w$
- ▶ capital rental:  $q$
- ▶ annuities...



# Annuities

- ▶ The problem: what to do with the wealth of households who die?
  - ▶ “accidental bequests”
- ▶ Assumption: households buy fair **annuities**.
- ▶ Each cohort  $\tau$  household gives  $a(t|\tau)$  to the insurance company.
- ▶ He gets paid:
  1. interest  $r(t)a(t|\tau)$
  2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)|t, \tau) = va(t|\tau) \quad (4)$$

- ▶ Effectively, the interest rate, conditional on survival, is  $r(t) + v$ .

# Firms

- ▶ A representative firm solves the standard problem.
- ▶ Factor prices are

$$q = f'(k)$$

$$w = f(k) - f'(k)k$$

# Equilibrium

## Definition

A CE is an allocation  $[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty}$  and a price system  $[w(t), q(t), r(t)]$  such that:

1.  $c(t|\tau)$  and  $a(t|\tau)$  solve the household's problem for cohort  $t - \tau$ .
2.  $w(t)$  and  $q(t)$  solve the firm's problem.
3. markets clear.
4. identities:  $L(t), C(t), r(t) = q(t) - \delta$

# Equilibrium

Market clearing:

- ▶ labor: implicit
- ▶ capital:  $K(t) = \int_0^t L(t|\tau) a(t|\tau) d\tau$ .
- ▶ goods: same as resource constraint.

Identities:

- ▶  $C(t) = \int_0^t L(t|\tau) c(t|\tau) d\tau$  etc

## Math Digression: Leibniz's Rule

We want to differentiate an integral

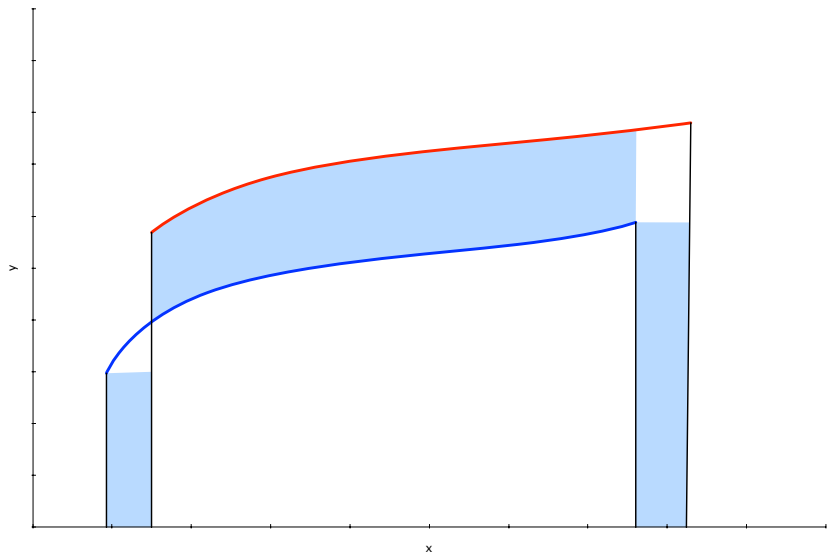
Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx \quad (5)$$

We have

$$\frac{\partial F}{\partial \theta} = f(b(\theta), \theta) b'(\theta) - f(a(\theta), \theta) a'(\theta) + \int_{a(\theta)}^{b(\theta)} f_{\theta}(x, \theta) dx \quad (6)$$

# Leibniz's Rule



# Households

The representative member of cohort  $\tau$  solves

$$\max \int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau)) dt$$

subject to

$$\dot{a}(t|\tau) = [r(t) + \nu]a(t|\tau) - c(t|\tau) + w(t) \quad (7)$$

## Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t) + v] - [\rho + v] = r(t) - \rho \quad (8)$$

budget constraint and TVC

$$\lim_{t \rightarrow \infty} D_{t,\tau} a(t|\tau) = 0 \quad (9)$$

where

$$D_{t,\tau} = \exp\left(-\int_{\tau}^t [r(z) + v] dz\right) \quad (10)$$

Notation:

- ▶  $D_{t,\tau}$  discounts a date  $t$  payment to  $\tau$ .
- ▶  $D_{\tau,t} = 1/D_{t,\tau}$  discounts a date  $\tau$  payment to  $t$ .



## Household: PIH

Claim: because of log utility, the household consumes a constant fraction of "human wealth:"

$$c(t|\tau) = (\rho + \nu) [a(t|\tau) + \omega(t)] \quad (11)$$

Human wealth is the present value of lifetime earnings

$$\omega(t) = \int_t^{\infty} D_{s,t} w(s) ds \quad (12)$$

Note: all persons alive at  $t$  have the same  $\omega$ .

## Proof: PIH

Present value budget constraint (see below):

$$\int_{\tau}^t D_{\tau,t} c(t|\tau) dt = a(t|\tau) + \omega(t) \quad (13)$$

Integrate Euler:

$$c(t|\tau) = c(\tau|\tau) \exp\left(\int_{\tau}^t [r(z) - \rho] dz\right) \quad (14)$$

Verify by differentiating and comparing with Euler

## Proof: PIH

Multiply both sides by  $D_{t,\tau}$ :

$$D_{t,\tau}c(t|\tau) = c(\tau|\tau) \exp\left(\int_{\tau}^t [r(z) - \rho - r(z) + v] dz\right) \quad (15)$$

$$= c(\tau|\tau) \exp(-[\rho - v][t - \tau]) \quad (16)$$

Present value of consumption

$$\int_{\tau}^{\infty} D_{t,\tau}c(t|\tau) = \frac{c(\tau|\tau)}{\rho - v} = a(\tau|\tau) + \omega(\tau) \quad (17)$$

Last step: show that the same holds for  $t > \tau$ ...

## Lifetime Budget Constraint

Claim:

$$D_{t,\tau}a(t|\tau) = a(\tau, \tau) + \int_{\tau}^t D_{z,\tau} [w(z) - c(z|\tau)] dz \quad (18)$$

Take  $\lim_{t \rightarrow \infty}$  and the LHS goes to 0 due to TVC.

Multiply by  $D_{\tau,t}$ :

$$a(t|\tau) = a(\tau|\tau)D_{\tau,t} + \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz \quad (19)$$

To verify, differentiate with respect to  $t$  and check that the flow budget constraint

$$\dot{a}(t|\tau) = [r(t) + v]a(t|\tau) - c(t|\tau) + w(t) \quad (20)$$

emerges.

## Lifetime Budget Constraint: Proof

$$\dot{a}(t|\tau) = a(\tau|\tau) \frac{\partial D_{\tau,t}}{\partial t} + D_{t,t} [w(t) - c(t|\tau)] + \int_{\tau}^t \frac{\partial D_{z,t} [w(z) - c(z|\tau)]}{\partial t} dz$$

and note that

1.  $\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t} [r(t) - v]$ , so that the first term becomes  $(r(t) + v) a(\tau|\tau) D_{\tau,t}$
2.  $D_{t,t} = \exp(0) = 1$ , so that the second term becomes  $w(t) - c(t|\tau)$
3. the 3rd term is

$$[r(t) + v] \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz = [r(t) + v] [a(t, \tau) - a(\tau|\tau) D_{\tau,t}]$$

Add all that up and the flow budget constraint emerges.

# Aggregation

$$c(t) = \int_{-\infty}^t L(t, \tau) c(t|\tau) d\tau / L(t) \quad (21)$$

$$= (\rho + \nu) [a(t) + \omega(t)] \quad (22)$$

- ▶ This is a form of **aggregation**: Aggregate consumption behaves like individual consumption.
  - ▶ As if a single individual made the choice.
- ▶ The budget constraint aggregates in the same way.
- ▶ How general is this?

# Equilibrium Dynamics

It would be tempting to say:

- ▶ Euler is unchanged relative to growth model
- ▶ Resource constraint is unchanged
- ▶ Everything behaves like the growth model

But this would be wrong:

- ▶ each person has an Euler equation that looks “standard”
- ▶ that does not mean that aggregate consumption also behaves that way

# Equilibrium Dynamics

- ▶ We have a system in  $c, a, \omega$ .
- ▶ Equations: consumption function, budget constraint, def of lifetime wealth:

$$c(t) = (\rho + v)[a(t) + \omega(t)]$$

$$\dot{a}(t) = (r(t) - (n - v))a(t) + w(t) - c(t)$$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(t) + v] dt\right) w(s) ds$$

- ▶ The strategy: Derive an Euler equation for aggregate consumption by differentiating the  $c(t)$  equation.



## Equilibrium Dynamics

- ▶ Differentiate the consumption function:

$$\dot{c} = (\rho + \nu) [\dot{a} + \dot{\omega}] \quad (23)$$

- ▶ Sub in budget constraint for  $\dot{a}$ .
- ▶ Differentiate def of  $\omega$  (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + \nu) \omega(t) - w(t) \quad (24)$$

- ▶ Sub that into  $\dot{c}$  and collect terms:

$$\dot{c}(t) = [r(t) - \rho] c(t) - (\rho + \nu) n a(t) \quad (25)$$

- ▶ Sub in  $k(t) = a(t)$  and the firm foc for  $r(t)$ :

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \quad (26)$$

## Intuition for $\omega(t)$

Think of human wealth as an asset with price  $\omega(t)$ .

Its instantaneous payoff consists of:

1. "dividend"  $w(t)$
2. capital gain  $\dot{\omega}(t)$

The asset price equals [required rate of return]  $\times$  [dividend + capital gain]

Required rate of return is  $r(t) + v$ .

$$[r(t) + v] \omega(t) = w(t) + \dot{\omega}(t) \quad (27)$$

## Note: Differentiating $\omega(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds \quad (28)$$

$\dot{\omega}(t)$  has 2 pieces:

1. Effect of changing lower bound of integral is integrand evaluated at  $s = t$ :  $w(t)$ .
2. Derivative of integrand w.r.to  $t$ :  
$$-[r(t) + \nu] \omega(t) = \int_t^\infty w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) ds.$$

Now note that

$$\frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) = \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) \times [-(r(t) + \nu)].$$

## Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu)n \frac{k(t)}{c(t)} \quad (29)$$

$$\dot{k} = f(k) - c - (n - \delta - \nu)k \quad (30)$$

with boundary conditions  $k(0)$  given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...

## Steady state

$$\dot{c} = 0 \implies c = \frac{(\rho + v)n}{f'(k) - \delta - \rho} k \quad (31)$$

Properties:

1.  $k \rightarrow 0 \implies c \rightarrow 0$  [as  $f' \rightarrow \infty$ ]
2.  $k \rightarrow k^{MGR}$  where  $f'(k^{MGR}) = \delta + \rho \implies c \rightarrow \infty$
3.  $c''(k) > 0$  [verify]

## Steady state

$$\dot{k} = 0 \implies c = f(k) - (n + \delta - \nu)k \quad (32)$$

Properties: as the standard growth model.

## Steady state

Solution for steady state  $k^*$

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \quad (33)$$

Unique steady state  $k^*$ :  $f(k)/k \searrow$  in  $k$ .  $-1/f'(k) \searrow$  in  $k$ .

# Dynamic efficiency

- ▶ **Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - \nu)k^* \quad (34)$$

$$f'(k_{GR}) - \delta = n - \nu \quad (35)$$

- ▶ Steady state:

$$f'(k^*) - \delta > \rho \quad (36)$$

[otherwise  $c/k < 0$ ]

- ▶ There can be overaccumulation relative to the Golden Rule.
- ▶ This happens when households are sufficiently impatient (high  $\rho$ ).
- ▶ Similar to the finite lifetime OLG model.



## Dynamic efficiency

- ▶ **Modified Golden Rule** for planner with discount factor  $\rho$  [effects of mortality and "annuities" cancel]:

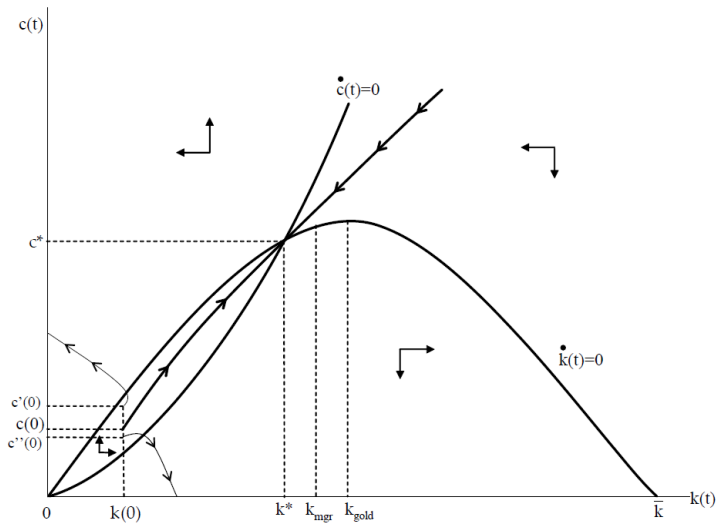
$$f'(k_{MGR}) - \delta = \rho \quad (37)$$

- ▶ Equilibrium avoids overaccumulation relative to MGR.
- ▶ This is not a robust feature of the model.
- ▶ Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- ▶ Example: labor efficiency declines with age.

## Dynamic efficiency

- ▶ Finite lifetimes are not necessary to generate overaccumulation.
- ▶ In this model, it is the presence of overlapping generations that destroys the welfare theorems.

# Phase diagram



## Phase diagram

- ▶ The dynamics closely resemble the growth model.
- ▶ A unique, globally saddle path stable steady state exists.
- ▶ Convergence is monotone.
- ▶ An analytically tractable model with OLG.

# Where Is This Used?

## Models of human capital

- ▶ combine the convenience of an infinitely lived decision maker
- ▶ capture that only young invest in education
- ▶ Akyol and Athreya (2005)

## Models of income / wealth distribution

- ▶ a version of perpetual youth: agents age stochastically
- ▶ Castaneda et al. (2003)

# Reading

- ▶ Acemoglu (2009), ch. 9.7-9.8.
- ▶ Blanchard and Fischer (1989), ch. 3.3

## References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Akyol, A. and K. Athreya (2005): “Risky higher education and subsidies,” *Journal of Economic Dynamics and Control*, 29, 979 – 1023.
- Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.
- Castaneda, A., J. Diaz-Gimenez, and J. V. Rios-Rull (2003): “Accounting for the US earnings and wealth inequality,” *Journal of political economy*, 111, 818–857.