Perpetual Youth Model

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Perpetual youth

- The standard growth model is very tractable.
- But it has an important limitation: all households are identical.
- For some questions, it is important to have households of **different ages**:
    - fiscal policies that redistribute across ages
    - models with life-cycle features: job search, matching, ...
- An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

Poisson Process

The Poisson process is the continuous time analog of i.i.d.

It is a counting process: it describes the distribution of the **number of events** occurring during a particular time interval.

$$\Pr(N(t) = n) = \frac{(\nu t)^n}{n!} e^{-\nu t} \tag{1}$$

The parameter $\nu > 0$ is the arrival rate:

$$\mathbb{E}\{N(t)\} = \lambda t \tag{2}$$

Mental image:

- randomly distribute points on a real line
- on average, there are λ points per unit length
- as time passes, move along the line and count the points

Poisson Process

The probability of an event over a short period ε is $\nu\varepsilon$.

- to show this: evaluate $\partial\mathbb{E}\{N(t+\varepsilon)|N(t)\}/\partial\varepsilon = \nu$

The probability of **no event** over a period of length τ is $\exp(-\nu\tau)$.

- the continuous time analogue of $(1-p)^t$

Demographics

- At $t = 0$, there are $L(0) = 1$ identical persons.
  - They are all newborns.
- At each instant, $nL(t)$ identical persons are born.
- Each person dies at each instant with **Poisson** probabilty ν.
- The population growth rate is $n - \nu > 0$:

$$L(t) = \exp([n - \nu]t) \tag{3}$$

Demographics

- The mass of persons at t aged $t-\tau$ is

$$\begin{aligned} L(t|\tau) &= \exp(-\nu(t-\tau)) \times n\exp((n-\nu)\tau) \\ &= \Pr(\text{live beyond } t-\tau)\ nL(\tau) \end{aligned}$$

- Notation: $x(t|\tau)$ means x at t for those born at τ.

Preferences

- Households are indexed by i.
- Conditional on surviving, households utility at date t is $e^{-\rho t}\ln(c_i(t))$.
- The probability of being alive after t "periods" is $\exp(-\nu t)$.
- Expected utility for date t is $e^{-\nu t}e^{-\rho t}\ln(c_i(t))$.
- Expected lifetime utility is

$$\int_0^\infty e^{-(\rho+\nu)t}\ln(c_i(t))\,dt \tag{4}$$

- Interesting: mortality simply increases the discount factor: $\rho+\nu$.

Technology

- The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- In per capita terms

$$\dot{k} = f(k) - c - (n - \nu + \delta) k \tag{5}$$

- $k = K/L$ is capital per capita and capital per worker.

Markets

Competitive markets for

- goods (numeraire)
- labor rental: w
- capital rental: q
- annuities...

Annuities

- The problem: what to do with the wealth of households who die?
  - "accidental bequests"
- Assumption: households buy fair **annuities**.
- Each cohort τ household gives $a(t|\tau)$ to the insurance company.
- He gets paid:
  1. interest $r(t)\,a(t|\tau)$
  2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)\,|t,\tau) = \nu a(t|\tau) \tag{6}$$

- Effectively, the interest rate, conditional on survival, is $r(t)+\nu$.

Firms

- A representative firm solves the standard problem.
- Factor prices are

$$\begin{aligned} q &= f'(k) \\ w &= f(k) - f'(k)k \end{aligned}$$

Equilibrium

Definition

A CE is an allocation $[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0,\tau\leq t}^{\infty}$ and a price system $[w(t), q(t), r(t)]$ such that:

1. $c(t|\tau)$ and $a(t|\tau)$ solve the household's problem for cohort $t-\tau$.
2. $w(t)$ and $q(t)$ solve the firm's problem.
3. markets clear.
4. identities: $L(t), C(t)$, $r(t) = q(t) - \delta$

Equilibrium

Market clearing:

- labor: implicit
- capital: $K(t) = \int_0^t L(t|\tau)\, a(t|\tau)\, d\tau$.
- goods: same as resource constraint.

Identities:

- $C(t) = \int_0^t L(t|\tau)\, c(t|\tau)\, d\tau$ etc

Math Digression: Leibniz's Rule

We want to differentiate an integral

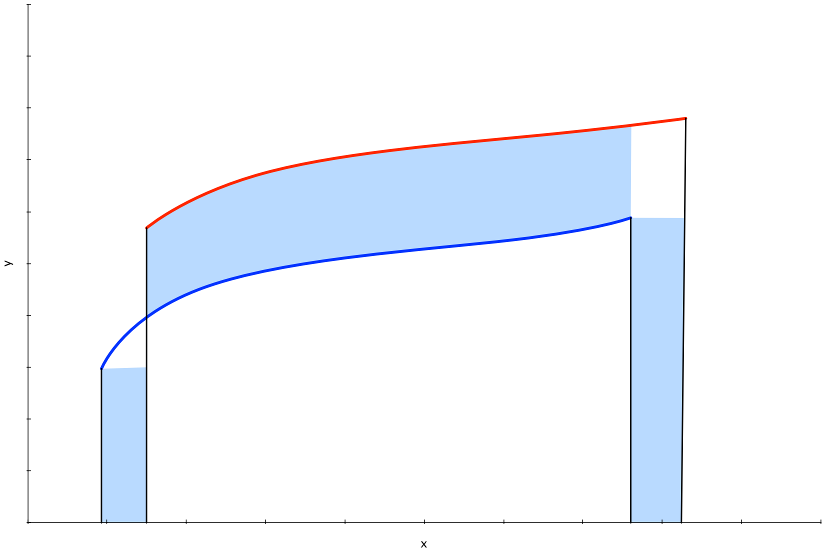
Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x,\theta)\, dx \tag{7}$$

We have

$$\frac{\partial F}{\partial \theta} = f(b(\theta),\theta)\, b'(\theta) - f(a(\theta),\theta)\, a'(\theta) + \int_{a(\theta)}^{b(\theta)} f_\theta(x,\theta)\, dx \tag{8}$$

Leibniz's Rule



Households

The representative member of cohort τ solves

$$\max \int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln\left(c\left(t|\tau\right)\right) dt$$

subject to

$$\dot{a}\left(t|\tau\right) = \left[r\left(t\right) + \nu\right] a\left(t|\tau\right) - c\left(t|\tau\right) + w\left(t\right) \tag{9}$$

Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t) + v] - [\rho + v] = r(t) - \rho \tag{10}$$

budget constraint and TVC

$$\lim_{t\to\infty} D_{t,\tau} a(t|\tau) = 0 \tag{11}$$

where

$$D_{t,\tau} = \exp\left(-\int_{\tau}^{t} [r(z) + v] dz\right) \tag{12}$$

Notation:

- $D_{t,\tau}$ discounts a date t payment to τ.
- $D_{\tau,t} = 1/D_{t,\tau}$ discounts a date τ payment to t.

Household: PIH

Claim: because of log utility, the household consumes a constant fraction of "human wealth:"

$$c(t|\tau) = (\rho + \nu)\left[a(t|\tau) + \omega(t)\right] \tag{13}$$

Human wealth is the present value of lifetime earnings

$$\omega(t) = \int_t^{\infty} D_{s,t} w(s)\, ds \tag{14}$$

Note: all persons alive at t have the same ω.

Proof: PIH

Present value budget constraint (see below):

$$\int_{\tau}^{t} D_{\tau,t} c(t|\tau) dt = a(t|\tau) + \omega(t) \tag{15}$$

Integrate Euler:

$$c(t|\tau) = c(\tau|\tau) \exp\left(\int_{\tau}^{t} [r(z) - \rho] dz\right) \tag{16}$$

Verify by differentiating and comparing with Euler

Proof: PIH

Multiply both sides by $D_{t,\tau}$:

$$D_{t,\tau}c(t|\tau) = c(\tau|\tau)\exp\left(\int_\tau^t [r(z) - \rho - r(z) + v]dz\right) \tag{17}$$
$$= c(\tau|\tau)\exp\left(-[\rho - v][t - \tau]\right) \tag{18}$$

Present value of consumption

$$\int_\tau^\infty D_{t,\tau}c(t|\tau) = \frac{c(\tau|\tau)}{\rho - v} = a(\tau|\tau) + \omega(\tau) \tag{19}$$

Last step: show that the same holds for $t > \tau$...

Lifetime Budget Constraint

Claim:

$$D_{t,\tau} a(t|\tau) = a(\tau,\tau) + \int_{\tau}^{t} D_{z,\tau} [w(z) - c(z|\tau)] dz \tag{20}$$

Take $\lim_{t\to\infty}$ and the LHS goes to 0 due to TVC.

Multiply by $D_{\tau,t}$:

$$a(t|\tau) = a(\tau|\tau) D_{\tau,t} + \int_{\tau}^{t} D_{z,t} [w(z) - c(z|\tau)] dz \tag{21}$$

To verify, differentiate with respect to t and check that the flow budget constraint

$$\dot{a}(t|\tau) = [r(t) + \nu] a(t|\tau) - c(t|\tau) + w(t) \tag{22}$$

emerges.

Lifetime Budget Constraint: Proof

$$\dot{a}(t|\tau) = a(\tau|\tau)\frac{\partial D_{\tau,t}}{\partial t} + D_{t,t}[w(t) - c(t|\tau)] + \int_\tau^t \frac{\partial D_{z,t}[w(z) - c(z|\tau)]}{\partial t} dz$$

and note that

1. $\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t}[r(t) - v]$, so that the first term becomes $(r(t) + v)\, a(\tau|\tau)\, D_{\tau,t}$
2. $D_{t,t} = \exp(0) = 1$, so that the second term becomes $w(t) - c(t|\tau)$
3. the 3rd term is

$$[r(t) + v]\int_\tau^t D_{z,t}[w(z) - c(z|\tau)]\, dz = [r(t) + v][a(t,\tau) - a(\tau|\tau) D_{\tau,t}]$$

Add all that up and the flow budget constraint emerges.

Aggregation

$$
\begin{aligned}
c(t) &= \int_{-\infty}^{t} L(t,\tau)c(t|\tau)d\tau / L(t) && (23) \\
&= (\rho+\nu)\left[a(t)+\omega(t)\right] && (24)
\end{aligned}
$$

- This is a form of **aggregation**: Aggregate consumption behaves like individual consumption.
  - As if a single individual made the choice.
- The budget constraint aggregates in the same way.
- How general is this?

Equilibrium Dynamics

It would be tempting to say:

- Euler is unchanged relative to growth model
- Resource constraint is unchanged
- Everything behaves like the growth model

But this would be wrong:

- each person has an Euler equation that looks "standard"
- that does not mean that aggregate consumption also behaves that way

Equilibrium Dynamics

- We have a system in c, a, ω.
- Equations: consumption function, budget constraint, def of lifetime wealth:

$$
\begin{aligned}
c(t) &= (\rho + \nu)\left[a(t) + \omega(t)\right] \\
\dot{a}(t) &= (r(t) - (n - \nu))\, a(t) + w(t) - c(t) \\
\omega(t) &= \int_t^{\infty} \exp\left(-\int_t^s \left[r(\iota) + \nu\right] d\iota\right) w(s)\, ds
\end{aligned}
$$

- The strategy: Derive an Euler equation for aggregate consumption by differentiating the $c(t)$ equation.

Equilibrium Dynamics

- Differentiate the consumption function:

$$\dot{c} = (\rho + \nu)[\dot{a} + \dot{\omega}] \tag{25}$$

- Sub in budget constraint for $\dot{a}$.
- Differentiate def of ω (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + \nu)\,\omega(t) - w(t) \tag{26}$$

- Sub that into $\dot{c}$ and collect terms:

$$\dot{c}(t) = [r(t) - \rho]\,c(t) - (\rho + \nu)\,na(t) \tag{27}$$

- Sub in $k(t) = a(t)$ and the firm foc for $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu)\,n\frac{k(t)}{c(t)} \tag{28}$$

Intuition for $\omega(t)$

Think of human wealth as an asset with price $\omega(t)$.

Its instantaneous payoff consists of:

1. "dividend" $w(t)$
2. capital gain $\dot{\omega}(t)$

The asset price equals [required rate of return] $\times$ [dividend + capital gain]

Required rate of return is $r(t) + \nu$.

$$[r(t) + v]\,\omega(t) = w(t) + \dot{\omega}(t) \tag{29}$$

Note: Differentiating $\omega(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu]\, d\iota\right)\, w(s)\, ds \tag{30}$$

$\dot{\omega}(t)$ has 2 pieces:

1. Effect of changing lower bound of integral is integrand evaluated at $s = t$: $w(t)$.
2. Derivative of integrand w.r.to t: $-[r(t) + \nu]\,\omega(t) = \int_t^\infty w(s)\, \frac{d}{dt} \exp(-\int_t^s [r(\iota) + \nu]\, d\iota)\, ds$.

Now note that
$\frac{d}{dt} \exp(-\int_t^s [r(\iota) + \nu]\, d\iota) = \exp(-\int_t^s [r(\iota) + \nu]\, d\iota) \times [-(r(t) + \nu)]$.

Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \tag{31}$$

$$\dot{k} = f(k) - c - (n - \delta - \nu) k \tag{32}$$

with boundary conditions $k(0)$ given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...

Steady state

$\dot{c} = 0 \Longrightarrow$

$$c = \frac{(\rho + \nu)\, n}{f'(k) - \delta - \rho} k \tag{33}$$

Properties:

1. $k \longrightarrow 0 \Longrightarrow c \longrightarrow 0$ [as $f' \longrightarrow \infty$]
2. $k \longrightarrow k^{MGR}$ where $f'\left(k^{MGR}\right) = \delta + \rho \Longrightarrow c \longrightarrow \infty$
3. $c''(k) > 0$ [verify]

Steady state

$\dot{k} = 0 \Longrightarrow$

$$c = f(k) - (n + \delta - \nu)k \tag{34}$$

Properties: as the standard growth model.

Steady state

Solution for steady state k^*

$$\frac{f(k^*)}{k^*} - (n - \nu + \delta) - \frac{(\rho + \nu) n}{f'(k^*) - \delta - \rho} = 0 \tag{35}$$

Unique steady state k^*: $f(k)/k \searrow$ in k. $-1/f'(k) \searrow$ in k.

Dynamic efficiency

- **Golden Rule** maximizes

$$
\begin{aligned}
c^* &= f(k^*) - (n + \delta - \nu) k^* && (36) \\
& f'(k_{GR}) - \delta = n - \nu && (37)
\end{aligned}
$$

- Steady state:

$$f'(k^*) - \delta > \rho \tag{38}$$

  [otherwise $c/k < 0$]
- There can be overaccumulation relative to the Golden Rule.
- This happens when households are sufficiently impatient (high ρ).
- Similar to the finite lifetime OLG model.

Dynamic efficiency

- **Modified Golden Rule** for planner with discount factor ρ [effects of mortality and "annuities" cancel]:

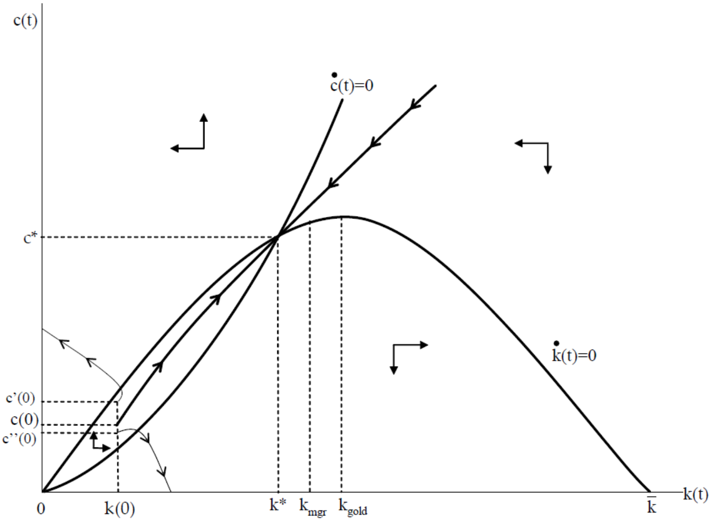
$$f'(k_{MGR}) - \delta = \rho \tag{39}$$

- Equilibrium avoids overaccumulation relative to MGR.
- This is not a robust feature of the model.
- Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- Example: labor efficiency declines with age.

Dynamic efficiency

- Finite lifetimes are not necessary to generate overaccumulation.
- In this model, it is the presence of overlapping generations that destroys the welfare theorems.

Phase diagram



Phase diagram

- The dynamics closely resemble the growth model.
- A unique, globally saddle path stable steady state exists.
- Convergence is monotone.
- An analytically tractable model with OLG.

Where Is This Used?

Models of human capital

- combine the convenience of an infinitely lived decision maker
- capture that only young invest in education
- Akyol and Athreya (2005)

Models of income / wealth distribution

- a version of perpetual youth: agents age stochastically
- Castaneda et al. (2003)

Reading

- Acemoglu (2009), ch. 9.7-9.8.
- Blanchard and Fischer (1989), ch. 3.3

References I

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