Problem Set 5: Growth Model in Continuous Time

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1 Continuous Time CIA Model. Cash and Credit Goods.

Demographics: A single representative household who lives forever.

Endowments: At the beginning of time, households hold k_0 units of capital and M_0 units of money.

Preferences:

$$\int_0^\infty e^{-\rho t} u(c_t, d_t) \, dt$$

where c and d are two consumption goods.

Technology:

$$f(k) = \dot{k} + c + d \tag{1}$$

Government: The government costlessly produces money M and hands it to households as lump-sum transfers. The money growth rate is constant at $g(M) = \gamma$. The government budget constraint is $\dot{M} = p\,x = g(M)\,M$ where p is the price level and x is the real lump-sum transfer.

Markets: There are competitive markets for goods and money. Households operate the technology (there are no firms).

CIA constraint: c has to be bought with cash:

$$c_t \leq M_t/p_t$$

d may be bought with credit.

Denote real balances by $m_t = M_t/p_t$.

Questions

1. Write down the household's Hamiltonian. Which are his states and controls? Derive first-order conditions for two cases: either the CIA constraint always binds or it never binds. Hint: the budget constraint is given by

$$\dot{k}_t + c_t + d_t + \dot{M}_t/p_t = f(k_t) + x_t$$

2. Define a competitive equilibrium.
3. Derive a set of equations that characterize the steady state. Show that the nominal interest rate equals zero, if the CIA constraint does not bind.
4. Determine the effects of a higher money growth rate on the steady state allocation. Assume that the utility function takes the form $u(c, d) = U(c) + V(d)$, where U and V are strictly concave functions.

2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt \tag{2}$$

where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t + \phi(\dot{m}) \tag{3}$$

where $\phi(\dot{m})$ is the cost of adjusting household money holdings. $\phi'(0) = 0$ and $\phi''(\dot{m}_t) > 0$.

Money: nominal money grows at exogenous rate $g(M)$. New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$.

Markets: money (numeraire), goods, capital rental (price q), labor (w).

Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t) \tag{4}$$

   where $r = q - \delta$. State the Hamiltonian. If you cannot figure this out, assume $\phi(\dot{m}) = 0$ and proceed (for less than full credit). Hint: Make m and k separate state variables.

2. State the first-order conditions.
3. Define a competitive equilibrium.
4. Characterize the steady state to the extent possible. What is the effect of a permanent change in $g(M)$?
5. What is the optimal rate of inflation? Explain.