The Growth Model: Discrete Time Competitive Equilibrium

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Competitive Equilibrium

We show that the CE allocation coincides with the planner's solution.

Model setup:

- Preferences, endowments, and technology are the same as before.
- Markets: goods, capital rental, labor rental

Households

A single representative household owns the capital and rents it to firms at rental rate q.

It supplies one unit of labor to the firm at wage rate w.

Preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

The budget constraint is:

$$k_{t+1} = (1-\delta)k_t + w_t + q_t k_t - c_t$$

Households: DP Representation

State variable: k.

Control: k'.

Bellman equation:

FOC

Envelope:

Euler equation:

$$u'(c) = \beta(1 + q' - \delta)u'(c')$$

Household: Solution

A pair of policy functions $c = \phi(k)$ and $k' = h(k)$ and a value function such that:

1. the policy functions solve the "max" part of the Bellman equation, given V;
2. the value function satisfies

In terms of **sequences**: $\{c_t, k_{t+1}\}$ that solve the Euler equation and the budget constraint.

The boundary conditions are k_0 given; and the transversality condition (TVC)

$$\lim_{t\to\infty} \beta^t u'(c_t) k_t = 0$$

Firms

- Firms rent capital and labor services from households, taking prices (q,w) as given.
- They maximize current period profits:

$$\max F(K,L) - wL - qK$$

- FOC

$$\begin{aligned} F_K(K,L) &= q \\ F_L(K,L) &= w \end{aligned}$$

Firms

- Assume constant returns to scale. Define

$$F(k^F)L = F(K,L)$$

- FOC's become

$$\begin{aligned} f'(k^F) &= q \\ f(k^F) - f'(k^F)k^F &= w \end{aligned}$$

- A **solution** is a pair (K,L) that satisfies the 2 FOC.

Equilibrium

An equilibrium is a sequence of
that satisfy:

Comparison with the Planner's Solution

One way of showing that the Planner's solution coincides with the CE is to appeal to the First and Second Welfare Theorems.

A more direct way is to show that the equations that characterize CE and the planner's solution are the same.

CE
$u'(c) = (1 + q' - \delta)\ \beta\ u'(c')$
$k' + c = f(k) + (1 - \delta)\ k$
$k' = (1 - \delta)k + w + qk - c$
$q = f'(k)$
$w = f(k) - f'(k)k$

Planner
$u'(c) = (f'(k') + 1 - \delta)\ \beta\ u'(c')$
$k' + c = f(k) + (1 - \delta)k$

Recursive Competitive Equilibrium

Recursive competitive equilibrium

Recursive CE is an alternative way of representing a CE that is more fully consistent with the DP approach.

- Everything is written as functions of the state variables.
- There are no sequences.

This is useful especially in models with

- heterogeneous agents where the distribution of households is a state variable;
- uncertainty, where we cannot assume that agents take future prices as given.

Recursive competitive equilibrium

Key feature of RCE

Everything in the economy is a function of the aggregate state S.
Individual agents often have additional states.

Agents form expectations using the law of motion for S: $S' = G(S)$

- E.g., to form expectations over future interest rate, use the law of motion for k and the price function $q = f'(k)$.

A fixed point problem:

- Agents' policy functions depends on the laws of motion.
- The laws of motion depend on agents' policy functions.

RCE in the example

The economy's *state variable* is aggregate K.

- Call its law of motion $K' = G(K)$.
- This is part of the equilibrium.

We solve the household problem for a saving function $k' = h(k, K)$.

- It depends on the private state k and the aggregate state K.

We solve the firm's problem for price functions $q(K), w(K)$.

Household

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = (1-\delta)k_t + w(K_t) + q(K_t)k_t - c_t$$

The household's problem has an individual state k and an aggregate state K.

Household

Bellman's equation is

$$\begin{aligned} V(k,K) &= \max u([1-\delta]k + w(K) + q(K)k - k') + \beta V(k',K') \\ K' &= G(K) \end{aligned}$$

Solution: $k' = h(k,K)$.

Firm

Nothing changes in the firm's problem.

Solution:

$$\begin{aligned} q(K) &= f'(K) \\ w(K) &= f(K) - f'(K)K \end{aligned}$$

RCE

Objects:

- household: a policy function $k' = h(k, K)$ and a value function $V(k, K)$.
- firm: price functions $q(K), w(K)$,
- law of motion for the aggregate state: $K' = G(K)$,

Equilibrium conditions:

- household: Given $G(K), q(K), w(K)$: the policy function solves the household's DP.
- firm: The price functions satisfy firm FOCs.
- Markets clear (same as before, except for notation).
- Household expectations are consistent with household behavior:

$$\boxed{h(K, K) = G(K)}$$

Consistency

$$\boxed{h(K,K) = G(K)}$$

Basic idea: expectations (governed by G) are consistent with actions.

In equilibrium, the household holds $k = K$ and chooses $k' = h(K,K)$.

He expects $K' = G(K)$.

Correct expectations requires $k' = K'$

Example: Heterogeneous Workers

Recursive CE: Example

Households

There are N_j households of type j.

The rest of the model is unchanged.

The representative type j household solves:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ s.t.\ k_{t+1} &= R_t k_t + w_t l_t - c_t \end{aligned}$$

Aggregate State

The aggregate state vector is the distribution of wealth:

$$\kappa = (\kappa_1, ..., \kappa_N) \tag{1}$$

κ_j is wealth of household j in equilibrium.

The household knows the law of motion

$$\kappa' = G(\kappa) \tag{2}$$

with jth element

$$\kappa'_j = G_j(\kappa) \tag{3}$$

Why not just $S = K$?

Aggregate State

Rule of thumb

With heterogeneous agents, the aggregate state includes the joint distribution of individual states.

Household Dynamic Program

$$
\begin{aligned}
V_j(k_j, \kappa) &= \max u(c_j, l_j) + \beta V_j\left(k_j', G(\kappa)\right) \\
k_j' &= R(\kappa) k_j + w(\kappa) l_j - c_j
\end{aligned}
$$

First-order conditions:

$$
\begin{aligned}
u_c(c_j, l_j) &= \beta V_{j,1}\left(k_j', G(\kappa)\right) && (4) \\
u_l(c_j, l_j) &= \beta V_{j,1}\left(k_j', G(\kappa)\right)\ w(\kappa) && (5)
\end{aligned}
$$

Envelope:

$$
V_{j,1}(k_j, \kappa) = u_c(c_j, l_j)\ R(\kappa) \quad (6)
$$

Household solution

A solution to the type j household problem consists of

- a value function V_j
- policy functions $k'_j = h_j(k_j, \kappa)$, $l_j = \ell_j(k_j, \kappa)$, and $c_j = g_j(k_j, \kappa)$.

These satisfy:

1. V_j is a fixed point of the Bellman equation, given h, ℓ and g.
2. h, ℓ and g "max" the Bellman equation.

Implicit: the household takes $S' = G(S)$ as given.

Firm

This is standard:

$$\max_{K,L} F(K,L) - w(\kappa)L - q(\kappa)K$$

FOC: Factor prices equal marginal products.

Solution: $K(\kappa)$ and $L(\kappa)$.

Market clearing

Goods:

$$F(K(\kappa), L(\kappa)) + (1-\delta) K(\kappa) = \sum_j N_j [g_j(\kappa_j, \kappa) + h_j(\kappa_j, \kappa)] \quad (7)$$

Labor:

$$L(\kappa) = \sum_j N_j \ell(\kappa_j, \kappa) \quad (8)$$

Capital:

$$K(\kappa) = \sum_j N_j \kappa_j \quad (9)$$

Note

Everything is either exogenous or a function of the state variables.

Recursive CE

Objects:

- household: functions V_j, h_j, ℓ_j, g_j
- firm: $K(\kappa), L(\kappa)$
- price functions: $w(\kappa), q(\kappa), R(\kappa)$
- law of motion: G.

These satisfy:

1. Household solution (4)
2. Firm first order conditions (2)
3. Market clearing (3 - 1 redundant)
4. Identity: $R(\kappa) = q(\kappa) + 1 - \delta$.
5. Consistency:
$$\kappa_j' = G_j(\kappa) = h_j(\kappa_j, \kappa) \;\; \forall j \tag{10}$$

Notes on RCE

All the objects to be found are functions, not sequences.

This helps when there are shocks:

- We cannot find the sequence of prices without knowing the realizations of the shocks.
- But we can find how prices evolve for each possible sequence of shocks.
- The price functions describe this together with the laws of motion for the states.

Notes on RCE

Functional analysis helps determine the properties of the policy functions and the laws of motion.

- E.g., we strictly concave utility we know that savings are increasing in k, continuous, differentiable, etc.

RCE helps compute equilibria.

- Find the household's optimal choices for every possible set of states.
- Then simulate household histories to find the laws of motion.

Example: Firms own capital

Example: Firms accumulate capital

The physical environment is the same as in the basic growth model.

Markets are now:

1. goods (numeraire)
2. labor rental (w)
3. shares of firms (q)
   supply of shares $= 1$
4. bonds (R)
   in zero net supply

Aggregate state: K with law of motion $K' = G(K)$

Household

The household also gets a share of the profits π.

$$V(a,b,K) = \max_{c,a',b'} u(c) + \beta V\left(a',b',G(K)\right) \tag{11}$$

subject to the budget constraint.

$$c + q(K)a' + b' = w(K) + [q(K) + \pi(K)]a + R(K)b \tag{12}$$

Decision rule $a' = g(a,b,K)$ and $b' = h(a,b,K)$
Verify that the two assets must pay the same rate of return:

$$R(G(K)) = \frac{q(G(K)) + \pi(G(K))}{q(K)} \tag{13}$$

Firm

Firms maximize the discounted present value of profits

$$W_0 = \max_{\{k_{t+1}, l_t\}} \sum_{t=0}^{\infty} \frac{\pi_t}{R_1 \times R_2 \times \ldots \times R_t} \tag{14}$$

- We see later: this is the same as maximizing firm value.

Period profits:

$$\pi = F(k,l) + (1-\delta)k - w(K)l - k' \tag{15}$$

Firm: Bellman Equation

Let's write the firm's problem in standard DP format:

- States: k, K
- Controls: k', l
- Current payoff: $\pi(k, l, k', K)$
- Law of motion: $k' = k'$
- Discount factor: $1/R(G(K))$

Bellman equation:

$$W(k, K) = \max_{k', l} \pi\left(k, l, k', K\right) + \frac{W(k', G(K))}{R(G(K))} \tag{16}$$

Firm: Bellman Equation

First-order conditions:

$$\frac{\partial \pi}{\partial l} = \frac{\partial F}{\partial l} - w(K) = 0 \tag{17}$$

$$\frac{\partial \pi}{\partial k'} = -1 = \frac{1}{R(G(K))} \frac{\partial W(.')}{\partial k'} \tag{18}$$

Envelope:

$$\frac{\partial W}{\partial k} = \frac{\partial \pi}{\partial k} = \frac{\partial F}{\partial k} + 1 - \delta \tag{19}$$

Implies:

$$\frac{\partial F}{\partial k} + 1 - \delta = R(G(K)) \tag{20}$$

Solution:

- Value function W
- Decision rules: $k'(k,K), l(k,K)$

Recursive Competitive Equilibrium

Objects:

1. Household: $V(a,b,K)$, $g(a,b,K)$, $h(a,b,K)$
2. Firm: $W(k,K)$, $k'(k,K)$, $l(k,K)$, $\pi(k,k',K)$
3. Price functions $w(K)$, $R(K)$, $q(K)$
4. Aggregate law of motion $G(K)$

RCE

Conditions:

1. Household optimization
2. Firm optimization
3. Market clearing
   3.1 bonds: $h(1,0,K) = 0$
   3.2 shares: $g(1,0,K) = 1$
   3.3 goods: RC
4. Consistency:
   4.1 $k'(K,K) = G(K)$
   4.2 $q(K) + \pi(K) = W(K,K)$: the share price is the present value of profits

The share price

The share price q has to deliver the same as the bond return R.

$$q(K) = \frac{q(K') + \pi(K')}{R(K')} \tag{21}$$

The firm value function W satisfies

$$W(K,K) = \pi(K) + \frac{W(K',K')}{R(K')} \tag{22}$$

Therefore, $q(K) + \pi(K) = W(K,K)$

That's the value of owning the firm in the current period.

Example: Heterogeneous Preferences

Model

Demographics:

- There are $j = 1, ..., J$ types of households.
- The mass of type j households is μ_j.
- The total mass is $\sum_j \mu_j = n$.

Preferences:

- $\max \sum_{t=0}^{\infty} \beta^t u_j(c_{jt})$.
- u_j is increasing and strictly concave and obeys Inada conditions.

Model

Technology: $F(k_t, n_t) + (1-\delta) n_t = C_t + k_{t+1}$

Endowments:

- Each household is endowed with one unit of labor in each period.
- At $t = 0$ household j is endowed with k_{j0} units of capital and with $b_{j0} = 0$ units of one period bonds.

Market arrangements are standard.

Household Problem

Nothing new here, except everything is indexed by j.

Define wealth as $a_{jt} = k_{jt} + b_{jt}$.

Impose no-arbitrage: $R = q + 1 - \delta$

Bellman equation:

Euler Equation:

$$u_j'(c) = \beta R' u_j'(c') \tag{23}$$

Solution (sequence language): $\{c_{jt}, a_{jt}\}$ that solve the Euler equation and budget constraint.

Boundary conditions: a_{j0} given and TVC

$$\lim_{t\to\infty} \beta^t u_j'(c_{jt}) a_{jt} = 0 \tag{24}$$

.

Competitive Equilibrium

A CE consists of sequences

which satisfy:

- 2 household conditions
- 2 firm first-order conditions (standard)
  $q_t = f'(k_t/n_t) + 1 - \delta$ and $w_t = f(k_t/n_t) - f'(k_t/n_t)\, k_t/n_t$
- Market clearing:

Steady State

- Similar to CE without time subscripts.
- Euler equation becomes:

$$\beta R = 1$$

- Interesting: we can find R without knowing preferences or wealth distribution.

Steady states with persistent inequality

Let's solve for steady state c_j as a function of prices and endowments (k_{j0}, b_{j0}).

Present value budget constraint

$$k_{j0} + b_{j0} = \sum_{t=0}^{\infty} \frac{c_j - w}{R^t} \tag{25}$$

In steady state:

$$k_{j0} + b_{j0} = \frac{c_j - w}{R - 1} \tag{26}$$

Key: all households have the same steady state **average propensity to consume** out of wealth.

Steady states with persistent inequality

Therefore: any distribution $\{k_{j0}, b_{j0}\}$ that sums to the steady state k yields

- the same aggregate consumption
- and therefore a steady state with permanent inequality

Therefore: redistributing assets leaves the economy in steady state

- with unchanged aggregate K

It would be harder to show that persistent inequality follows from *any* initial asset distribution which features capital inequality.

Lump-sum Taxes

Impose a lump-sum tax τ on type j households.

Throw revenues into the ocean.

How does the steady state change?

- New present value budget constraints:

$$k_{j0} + b_{j0} = \frac{c_j - w - \tau_j}{R - 1} \tag{27}$$

- Euler equation unchanged
- Therefore R unchanged.
- Households simply cut consumption by τ_j.
- All aggregates unchanged.

Differences in β

- Now imagine households differ in their β's, but not in their u functions.
- For simplicity, assume that $u(c) = c^{1-\sigma}/(1-\sigma)$.
- What would the asset distribution look like in the limit as $t \to \infty$?

Interesting Applications of Growth Models

Understanding hours worked:

- how elastic is labor supply: Bick et al. (2022), Keane and Rogerson (2012)
- labor supply and taxation: Ohanian et al. (2008)

Business cycles:

- Chari et al. (2007)

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Krusell (2014), ch. 5 on Recursive Competitive Equilibrium.
- Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey et al. (1989), ch. 1 is a nice introduction.

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