Two Sector Models

Prof. Lutz Hendricks

Econ720

August 22, 2018

Two Sector Models

We relax the assumption that there is only one good at each date.

There are no major changes in methods.

Multi-sector models are used to study issues such as:

- technical change that is "embodied" in capital goods,
- human capital,
- international trade.

The Environment

Demographics:

- There is a unit mass of **households** who live forever.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

- v is work; $1 - v$ is leisure.

Technologies

Consumption goods are produced according to

$$Y_1 = F(K_1, L_1)$$

and capital goods according to

$$Y_2 = G(K_2, L_2)$$

The resource constraints are

$$\begin{aligned} L_{1t} + L_{2t} &= \nu_t \\ K_{1t} + K_{2t} &= K_t \\ Y_{1t} &= c_t \\ Y_{2t} &= K_{t+1} - (1-\delta)K_t \end{aligned}$$

Planning Problem

The **planner** maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

subject to the resource constraints.

Since capital can be costlessly reallocated between sectors, the state variable is K_t.

The controls $c_t, L_{1t}, L_{2t},$ and φ_t.

φ_t is the fraction of capital employed in sector 1

$$\begin{aligned} K_{1t} &= \varphi_t \, K_t \\ K_{2t} &= (1 - \varphi_t) K_t \end{aligned}$$

Exercise

Suppose that reallocating capital across sectors is not possible.

Solve the planner's problem.

We now need 2 states (K_{1t}, K_{2t}).

Planning Problem

The Bellman equation is

$$\begin{aligned} V(K) &= \max u(F(\varphi K, L_1), 1 - L_1 - L_2) \\ &\quad + \beta V(K(1-\delta) + G([1-\varphi]K, L_2)) \end{aligned}$$

where the choice variables are L_1, L_2, φ.

Planning Problem

FOCs:

$$\begin{aligned} u_l &= \beta V'(K')G_L = u_c F_L \\ u_c F_K &= \beta V'(K')G_K \end{aligned}$$

Envelope:

$$\begin{aligned} V'(K) &= \varphi F_K u_c + \beta V'(K')\{1-\delta+(1-\varphi)G_K\} \\ &= u_c F_K + (1-\delta)\beta V'(K') \end{aligned}$$

Interpretation...

Planning Problem

Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.') \frac{F_K(.')}{G_K(.')} \{1 - \delta + G_K(.')\}$$

Static conditions

$$F_K / F_L = G_K / G_L$$
$$u_l = u_c F_L$$

Interpretation below...

Solution: Planning Problem

Sequences $\{c_t, v_t, K_{t+1}, \varphi_t, L_{1t}, L_{2t}\}$ that satisfy:

- 3 FOCs;
- 3 feasibility conditions;
- TVC: $\lim_{t\to\infty} \beta^t \, u_c(t) \, K_t = 0$.
- K_0 given.

Intuition: Static condition

$F_K/F_L = G_K/G_L$

The static condition equates marginal rates of substitution in the two sectors.

This is necessary for maximizing output for given inputs.

Intuition: Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.') \frac{F_K(.')}{G_K(.')} \{1 - \delta + G_K(.')\}$$

Consider first the case $F_K = G_K$.

Then we get the conventional Euler equation

$$u_c = \beta \ u_c \left(.'\right) \ \left\{1 - \delta + G_K \left(.'\right)\right\}$$

Intuition: Euler equation - General case

- At any point in time, consumption can be converted into next period capital at a marginal rate of transformation G_K/F_K.
- Period t: Convert 1 unit of c into G_K/F_K units of K'.
- Period $t+1$: Produce an additional

$$\{(1-\delta)+G_K(.')\}G_K/F_K$$

  units of date t+2 capital.
- Convert the additional date t+2 capital into date t+1 consumption at the rate of transformation

$$F_K(.')/G_K(.')$$

- Eat this. This leaves all variables after $t+1$ unchanged.

Planner: Steady State

In steady state, the Euler equation simplifies to

$$\beta\{1-\delta+G_K\}=1$$

Because the MRT, G_K/F_K, is constant this is the same as in the one sector model.

Competitive Equilibrium

Competitive Equilibrium

Watch your units!

Notation

- P_i are the prices of the goods.
- $p_2 = P_2/P_1$.
- RP_1 and wP_1 are the rental prices of capital and labor.

Competitive Equilibrium

Consumption sector **firms** maximize period profits:

$$\max Y_1 - RK_1 - wL_1$$

The FOCs are as usual:

$$\begin{aligned} R &= F_K \\ w &= F_L \end{aligned}$$

Competitive Equilibrium

Capital sector firms:

$$\max P_2 Y_2 - P_1 R K_2 - P_1 w L_2$$

Divide through by P_1 to obtain

$$\max p_2 Y_2 - R K_2 - w L_2$$

The FOCs are

$$\begin{aligned} R/p_2 &= G_K \\ w/p_2 &= G_L \end{aligned}$$

Households

The budget constraint is

$$P_{2t}k_{t+1} = P_{2t}(1-\delta)k_t + P_{1t}R_tk_t + P_{1t}(w_tv_t - c_t)$$

Divide through by P_1 to obtain the budget constraint in real terms:

$$p_{2t}k_{t+1} = (1-\delta)p_{2t}k_t + R_tk_t + w_tv_t - c_t$$

Rate of return

At t: Give up p_{2t} units of consumption and buy $dk_{t+1} = 1$

At $t+1$:

- Receive rental income $R_{t+1} dk_{t+1}$.
- Sell the undepreciated capital: $(1-\delta) p_{2,t+1} dk_{t+1}$.

The rate of return is

$$\begin{aligned} 1 + r_{t+1} &= \frac{R_{t+1} + (1-\delta) p_{2,t+1}}{p_{2,t}} \\ &= R_{t+1}/p_{2,t} + (1-\delta)\pi_{t+1} \end{aligned}$$

$\pi_{t+1} \equiv p_{2,t+1}/p_{2,t}$ is the price appreciation of k.

Household problem

$$V(k) = \max_{k',v} u\left(wv + Rk + (1-\delta)p_2 k - p_2 k', 1 - v\right) \tag{1}$$

$$+\beta V\left(k'\right) \tag{2}$$

FOCs:

$$u_1 w = w_2 \tag{3}$$

$$u_1 p_2 = \beta V'\left(k'\right) \tag{4}$$

Envelope:

$$V'(k) = u_1 \times (R + (1-\delta)p_2) \tag{5}$$

Euler:

$$u_1 = \beta u_1\left(.'\right) \frac{R' + [1-\delta]p_2'}{p_2} \tag{6}$$

Market clearing

- Labor: $L_{1t} + L_{2t} = \nu_t$.
- Capital: $K_{1t} + K_{2t} = K_t = a_t / p_{2t}$.
- Goods:

$$
\begin{aligned}
Y_{1t} &= c_t \\
Y_{2t} &= K_{t+1} - (1-\delta) K_t
\end{aligned}
$$

Equilibrium Definition

A CE is a sequence of prices

and quantities

which satisfy (12 equations in 11 unknowns):

Equilibrium prices

The firms' FOCs imply that

$$R = p_2 G_K = F_K$$

and therefore

$$p_2 = F_K / G_K$$

In words: the relative price equals the marginal rate of transformation.

Exercise: Show that the solutions of the planning problem and the CE coincide by substituting prices for derivatives of F and G in the planner's FOCs.

One-sector Reduced Form

A One-sector Reduced Form

- We can construct a two sector model that looks very much like a one sector model.
- This requires the assumption

$$G(K,L) = AF(K,L)$$

for some constant A

A One-sector Reduced Form

- Then static optimality

$$F_K/F_L = G_K/G_L$$

implies

$$k_1 = k_2$$

where $k = K/L$.

- The relative price of capital is constant

$$p_2 = 1/A$$

A One-sector Reduced Form

- We can write a single **aggregate resource constraint**:
- Define aggregate real output as

$$
\begin{aligned}
Y &= Y_1 + Y_2/A \\
&= F(K_1, L_1) + F(K_2, L_2) \\
&= (L_1 + L_2) f(k) \\
&= F(K, L) \\
&= c + (K_{t+1} - [1-\delta]K_t)/A
\end{aligned}
$$

- Choose units of capital such that $A = 1$: $\tilde{K} = K/A$.
- Then the resource constraint looks like a one sector model:

$$Y_t = c_t + \tilde{K}_{t+1} - (1-\delta)\tilde{K}_t = F(\tilde{K}_t, L_t)$$

Why is this useful?

We can write down a model with cross-country (or cross-industry) productivity differentials without having to construct a full-blown multi-sector model with endogenous prices.

In the data, the relative **price of capital** varies greatly across countries. We can model that.

We can study **investment specific technical change**.

- Assume that A grows at some rate.
- Then the relative price of capital falls over time (as it does in the data).
- The model generates an evolution of the industrial structure (e.g. movement from ag to industry).
- Greenwood, Hercowitz, and Krusell (1997) find that such technical change accounts for 60 percent of overall productivity growth.

Summary

- Nothing fundamental changes when there are multiple sectors.
- The main additional complexity is in the household budget constraint because there may be capital gains terms.
- The dynamics of two sector models is much more complex than that of one sector models.