

## 1 A Planning Problem

The economy is populated by a unit mass of infinitely lived households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t u(c_{Mt}, c_{Ht})$$

where  $c_{jt}$  denotes consumption of good  $j$ . The household has a unit time endowment in each period.

There are two goods in the economy, indexed by  $j = M, H$ . The production function for good  $M$  is  $F(k_{Mt}, h_{Mt})$ ; it is used for investment and consumption ( $c_{Mt}$ ). The production function for good  $H$  is  $G(k_{Ht}, h_{Ht})$ ; it is consumed as  $c_{Ht}$ .  $k_{jt}$  denotes capital input in sector  $j$  and  $h_{jt}$  denotes labor input. Capital goods depreciate at the common rate  $\delta$ .

(a) Assume that capital cannot be moved between sectors. Once installed in sector  $j$  it stays there forever. Formulate the Dynamic Programming problem solved by a central planner.

(b) For the remainder of the question assume that capital can be moved freely between sectors. Formulate the planner's Dynamic Program.

(c) Define a solution to the Planner's problem.

### 1.1 Answer Sketch: Planning Problem

(a) The planner solves (in sequence language):

$$\max \sum_{t=0}^{\infty} \beta^t u(c_{Mt}, c_{Ht})$$

subject to

$$\begin{aligned} c_{Ht} &= G(k_{Ht}, h_{Ht}) \\ k_{jt+1} &= (1 - \delta) k_{jt} + i_{jt} \\ i_{jt} &\geq 0 \\ c_{Mt} + i_{Mt} + i_{Ht} &= F(k_{Mt}, h_{Mt}) \end{aligned}$$

There are other ways of writing this. The state variables are both capital stocks. The Dynamic Program is therefore:

$$V(k_M, k_H) = \max u(F(k_M, h_M) - i_M - i_H, G(k_H, h_H)) + \beta V((1 - \delta) k_M + i_M, (1 - \delta) k_H + i_H)$$

subject to  $i_j \geq 0$ .

(b) The constraint set changes if capital can be moved between sectors. Effectively, the non-negativity constraints on investment are dropped. But it is then more convenient to write the constraints as

$$\begin{aligned} c_{Ht} &= G(k_{Ht}, h_{Ht}) \\ k_{t+1} &= (1 - \delta) k_t + F(k_t - k_{Ht}, 1 - h_{Ht}) - c_{Mt} \end{aligned}$$

The Dynamic Programming problem is now

$$V(k) = \max u[(1 - \delta) k + F(k - k_H, 1 - h_H) - k', G(k_H, h_H)] + \beta V(k')$$

(c) The first order conditions are

$$u_M F_k = u_H G_K \tag{1}$$

$$u_M F_H = u_H G_H \tag{2}$$

$$u_M = \beta V'(k') \tag{3}$$

The envelope condition is

$$V'(k) = u_M [(1 - \delta) + F_K]$$

Combining the last 2 equations yields the standard Euler equation

$$u_M = \beta u_M(.) [(1 - \delta) + F_K(.)] \quad (4)$$

A solution to the planner's problem (in sequence language) consists of sequences  $\{k_t, k_{Ht}, c_{Mt}, c_{Ht}\}$  which solve the first-order conditions (1) through (4) and the constraint  $c_{Ht} = G(k_{Ht}, h_{Ht})$ .

## 2 Consumption Taxes in a Growth Model

Consider the following version of the growth model. There is a single representative agent with preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

where  $c_t$  is consumption in period  $t$ , and  $0 < \beta < 1$ . The worker is endowed with one unit of time in each period but does not value leisure.

There are two production sectors. One sector produces the consumption good using a Cobb-Douglas technology:

$$c_t = k_{ct}^\theta n_{ct}^{1-\theta}$$

where  $k_{ct}$  and  $n_{ct}$  are capital and labor inputs to this sector at time  $t$  respectively. The other sector produces capital goods also using a Cobb-Douglas technology:

$$i_t = A k_{it}^\eta n_{it}^{1-\eta}$$

where  $k_{it}$  and  $n_{it}$  are capital and labor inputs to the investment sector. Feasibility requires:

$$\begin{aligned} (1 - \delta)k_t + i_t &= k_{t+1} \\ k_{ct} + k_{it} &= k_t \\ n_{ct} + n_{it} &= 1 \end{aligned}$$

where  $\delta$  is the depreciation rate for physical capital. Thus, we are assuming that capital is completely mobile across sectors. The initial capital stock  $k_0$  is given.

- Define a competitive equilibrium for this economy in sequence form.
- Define a steady state competitive equilibrium for this economy. Derive an equation to characterize the steady state value of the capital stock.
- Assume that the government places a proportional tax on consumption expenditures equal to  $\tau_c$  and then simply throws away the tax revenues. How will this affect the steady state values for the capital stock, investment and consumption? Justify your answer.

### 2.1 Answer Sketch: Consumption Tax

- The numeraire is capital. The price of consumption is  $p_t$ . The household maximizes discounted utility subject to

$$k_{t+1} = R_{t+1} k_t + w_t - p_t c_t$$

The Euler equation is

$$u'(c_t) = \beta R_{t+1} u'(c_{t+1}) p_t / p_{t+1}$$

Firms in sector  $j$  solve

$$\max p_j F(k_j, n_j) - r k_j - w n_j$$

First order conditions are

$$\begin{aligned} w/p_j &= f(x_j) - f'(x_j) x_j \\ r/p_j &= f'(x_j) \\ x_j &= k_j/n_j \end{aligned}$$

Competitive Equilibrium: Sequences  $\{c_t, k_t, k_{jt}, n_{jt}, R_t, r_t, w_t, p_t\}$  which satisfy:

2 household conditions

4 firm conditions

Market clearing: Labor.  $c = k_c^\theta n_c^{1-\theta}$ .  $k_{+1} = A k_i^\eta n_i^{1-\eta} + (1 - \delta) k$ .

Identities:  $k = k_i + k_c$ .  $R = 1 + r - \delta$ .

(b) Steady state: A steady state consists of the same 10 variables (without the time subscripts), which satisfy the same 11 conditions. The Euler equation becomes  $\beta R = 1$ . The investment firm's FOC determines the capital-labor ratio in that sector:

$$r = R - 1 + \delta = A \eta x_i^{\eta-1}$$

The market clearing condition for good  $i$  implies:

$$\delta k = A k_i^\eta n_i^{1-\eta}$$

The requirement that  $w/r$  is the same in both sectors yields

$$x_c \frac{1 - \theta}{\theta} = x_i \frac{1 - \eta}{\eta}$$

Together with

$$k = k_i + k_c = n_i x_i + (1 - n_i) x_c$$

we have an equation solving for  $n_i$ :

$$k = A n_i x_i^\eta / \delta = n_i x_i + (1 - n_i) x_c$$

The solution is

$$n_i = x_i^{1-\eta} \delta / A$$

Hence,  $k = x_i$ .

(c) Consumption tax: The only change is in the household budget constraint, where prices are replaced with  $(1 + \tau) p_t$ . This does not affect the Euler equation or any of the other equations used in the derivation of the steady state value of  $k$ . The only change is that consumption falls by the amount of the tax.

### 3 Two technologies

Consider an economy with a large number of infinitely lived identical households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t \log c_t.$$

Each household is endowed with  $k_0$  units of capital in period 0 and 1 unit of labor each period. The number of households in period  $t$  is  $N_t$ , where  $N_{t+1} = \eta N_t$ ,  $\eta > 1$ . For simplicity, assume that  $N_0 = 1$ .

We will consider two alternative technologies for this economy:

Technology 1:

$$Y_t = \gamma^t K_t^\theta N_t^{1-\theta}$$

Technology 2:

$$Y_t = \gamma^t K_t^\mu N_t^\phi L_t^{1-\mu-\phi}$$

In these technologies,  $\gamma > 1$  is the rate of exogenous total factor productivity growth,  $K_t$  is total (*not* per capita) capital,  $Y_t$  is total output, and  $L_t$  is the total stock of land. Land is assumed to be a fixed factor; it can not be produced and does not depreciate. To simplify without loss of generality, assume that  $L_t = 1$  for all  $t$ .

The resource constraint, assuming 100% depreciation of capital each period is given by

$$N_t c_t + K_{t+1} \leq Y_t;$$

with  $K_0 = k_0$  given.

1. Suppose that the only technology available is the first one.
  - (a) Formulate, as a dynamic programming problem, the social planner's problem that weights all individuals utility equally. That is, the planner weights utility in period  $t$  by the number of identical agents alive in that period.
  - (b) Characterize the balanced growth path of this economy. ("Characterize" means that you must derive a set of equations that determines all endogenous variables along this path. You do not need to solve these equations.) Solve explicitly for the growth rate of per capita consumption ( $c_t$ ) along this path.
2. Repeat part 1 using the second technology in place of the first.
3. Compare how the rate of population growth  $\eta$  affects the rate of per capita growth in the two cases. Provide an explanation for your findings.

### 3.1 Answer: Two technologies<sup>1</sup>

1a. The sequence problem of the social planner is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} N_t \beta^t \log c_t,$$

subject to

$$N_t c_t + K_{t+1} = \gamma^t K_t^\theta N_t^{1-\theta}, \quad K_0 \text{ given.}$$

and

$$N_{t+1} = \eta N_t = \eta (\eta^t N_0) = \eta^{t+1}.$$

The planner's problem can be rewritten as

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta\eta)^t \log c_t,$$

subject to

$$c_t = \gamma^t \left( \frac{K_t}{\eta^t} \right)^\theta - \frac{K_{t+1}}{\eta^t}; \quad K_0 \text{ given.} \tag{5}$$

The dynamic program of the social planner is then

$$V(K, t) = \max_{K'} \left\{ \log \left[ \gamma^t \left( \frac{K}{\eta^t} \right)^\theta - \frac{K'}{\eta^t} \right] + \beta\eta V(K', t') \right\},$$

subject to  $t' = t + 1$ .

1b. To characterize the balanced growth path of this economy:

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<sup>1</sup>Due to Joydeep Bhattacharya.

$$\begin{aligned} \text{Foc w.r.t. } K' & : \frac{1}{\eta^t c} = \beta \eta V_1(K', t'), \\ \text{EC w.r.t } K & : V_1(K, t) = \frac{1}{c} \theta \left( \frac{\gamma}{\eta} \right)^t \left( \frac{K}{\eta^t} \right)^{\theta-1}. \end{aligned}$$

Combining the two obtains the Euler equation:

$$\frac{1}{\eta^t c} = \frac{\beta}{c'} \theta \left( \frac{\gamma}{\eta} \right)^{t'} \left( \frac{K'}{\eta^{t'}} \right)^{\theta-1},$$

or (reverting to the time notation)

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \theta \gamma^{t+1} \left( \frac{K_{t+1}}{\eta^{t+1}} \right)^{\theta-1}. \quad (6)$$

Along a balanced growth path (per capita variables grow at a constant rate, say  $g$  – remember that  $K_t$  is total capital stock):

$$c_t = g^t \bar{c}; \quad (7)$$

$$K_t = (g\eta)^t \bar{K} \quad (8)$$

This, with (6), implies:

$$1 = \frac{\beta}{g} \theta \gamma^{t+1} (g^{t+1} \bar{K})^{\theta-1}. \quad (9)$$

The resource constraint (5) on the balanced growth path

$$g^t \bar{c} = \gamma^t \left( \frac{(g\eta)^t \bar{K}}{\eta^t} \right)^\theta - \frac{(g\eta)^{t+1} \bar{K}}{\eta^t} \Rightarrow \quad (10)$$

$$(\gamma g^\theta)^t \bar{K}^\theta = g^t (\bar{c} + g\eta \bar{K}) \quad (11)$$

Equations (7) - (11) characterize the balanced growth path of this economy.

Observe that (11) can hold for all  $t$  is iff  $\bar{K}^\theta = \bar{c} + g\eta \bar{K}$  and

$$\gamma g^\theta = g \Leftrightarrow g = (\gamma)^{\frac{1}{1-\theta}}$$

We could have arrived at the same result by using (9), which will hold for all  $t$  iff

$$g = (\gamma)^{\frac{1}{1-\theta}} \text{ and } \bar{K} = \left( \frac{\beta \theta}{g} \right)^{\frac{1}{1-\theta}}$$

2a. The sequence problem of the social planner is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} N_t \beta^t \log c_t,$$

subject to

$$N_t c_t + K_{t+1} = \gamma^t K_t^\mu N_t^\phi, \quad K_0 \text{ given.}$$

and

$$N_{t+1} = \eta N_t = \eta (\eta^t N_0) = \eta^{t+1}.$$

The planner's problem can be rewritten as

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta \eta)^t \log c_t,$$

subject to

$$c_t = \gamma^t K_t^\mu (\eta^t)^{\phi-1} - \frac{K_{t+1}}{\eta^t}; K_0 \text{ given.} \quad (12)$$

The dynamic program of the social planner is then

$$V(K, t) = \max_{K'} \left\{ \log \left[ \gamma^t K^\mu (\eta^t)^{\phi-1} - \frac{K'}{\eta^t} \right] + \beta \eta V(K', t') \right\},$$

subject to  $t' = t + 1$ .

2b. To characterize the balanced growth path of this economy:

$$\begin{aligned} \text{Foc w.r.t. } K' &: \frac{1}{\eta^t c} = \beta \eta V_1(K', t'), \\ \text{EC w.r.t } K &: V_1(K, t) = \frac{1}{c} \mu \gamma^t K^{\mu-1} (\eta^t)^{\phi-1}. \end{aligned}$$

Combining the two obtains the Euler equation:

$$\frac{1}{\eta^t c} = \frac{\beta \eta}{c'} \mu \gamma^{t'} (K')^{\mu-1} (\eta^{t'})^{\phi-1},$$

or (reverting to the time notation)

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}} \mu \gamma^{t+1} (K_{t+1})^{\mu-1} (\eta^{t+1})^\phi. \quad (13)$$

Once again, on the balanced growth path:

$$c_t = g^t \bar{c}; \quad (14)$$

$$K_t = (g\eta)^t \bar{K}. \quad (15)$$

This, with (13), gets

$$1 = \frac{\beta}{g} \mu \gamma^{t+1} \left( (g\eta)^{t+1} \bar{K} \right)^{\mu-1} (\eta^{t+1})^\phi. \quad (16)$$

The resource constraint (5) on the balanced growth path:

$$g^t \bar{c} = \gamma^t \left( (g\eta)^t \bar{K} \right)^\mu (\eta^t)^{\phi-1} - \frac{(g\eta)^{t+1} \bar{K}}{\eta^t} \Rightarrow \quad (17)$$

$$(\gamma g^\mu \eta^{\phi+\mu-1})^t \bar{K}^\mu = g^t (\bar{c} + g\eta \bar{K}). \quad (18)$$

Equations (7) - (11) characterize the balanced growth path of this economy.

Observe that (11) can hold for all  $t$  iff

$$\bar{K}^\mu = \bar{c} + g\eta \bar{K} \text{ and} \quad (19)$$

$$g = \gamma g^\mu \eta^{\phi+\mu-1} \Leftrightarrow g = \left( \frac{\gamma}{\eta^{1-\phi-\mu}} \right)^{\frac{1}{1-\mu}} \quad (20)$$

Once again, one can arrive at the same result by using (16), which will hold only if  $g$  is as above and

$$\bar{K} = \left( \frac{\beta \mu}{g} \right)^{\frac{1}{1-\mu}}$$

3. In the first case  $\eta$  does not affect  $g$ . In the second,  $g$  is inversely proportional to  $\eta$ : a higher population growth rate reduces the growth rate of per capita variables in the economy (it is even possible that  $g < 1$ ). With the first technology, the economy accumulates capital on a balanced growth path consistent with the growth of *enhanced* labor. One can think of the productivity growth as labor-enhancing (i.e., labor efficiency growing at the rate of

$\gamma^{\frac{1}{1-\theta}}$ ) and accordingly the capital accumulation takes both population growth and labor productivity growth into account (and grows at the rate of  $\eta \gamma^{\frac{1}{1-\theta}}$ ). As a result, per capita output grows at the rate  $\gamma^{\frac{1}{1-\theta}}$ .

With the second technology, the third factor, land, is fixed. As before  $\gamma^{\frac{1}{1-\mu}}$  can be accounted for both labor- and land-enhancing productivity growth. Here, the population as before grows at the rate  $\eta$  and the aggregate capital stock can be made to grow enough to provide for the growing population (i.e., grow at  $\eta g$ ), but the land is fixed. The growth rate of capital then must be adjusted by a factor of population, so that per capita output also grows at  $g$ . This is achieved by (20).