Review Questions: Infinite Horizon Models in Discrete Time

Econ720. Fall 2016. Prof. Lutz Hendricks

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1 Taxes and government spending

Consider the following growth economy, extended to include government spending and a tax on output at rate τ_t. A representative household chooses $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to solve the problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} = (1 - \tau_t) f(k_t)$$

k_0 given.

Capital depreciates completely each period. The government finances exogenous spending, g_t, each period by levying taxes at rate τ_t. Government spending does not affect private utility or production possibilities. The government budget constraint is

$$\tau_t f(k_t) = g_t$$

You may find it convenient to define government spending as a share of output by introducing the variable

$$s_t^g = g_t / f(k_t)$$

Further, note the two budget constraints imply the aggregate resource constraint

$$c_t + g_t + k_{t+1} = f(k_t)$$

or

$$c_t + k_{t+1} = (1 - s_t^g) f(k_t)$$

The household takes the time paths of the policy variables as given. Assume the functional forms: $u(c) = \ln(c)$ and $f(k) = k^\theta$, $0 < \theta < 1$.

(a) Show that there is a maximum sustainable capital stock, k_{max}, for this economy.

(b) Assuming that $k_0 \in (0, k_{\max})$ find the steady state level of the capital stock. Assume that τ is constant over time. Note that there are no firms; households produce and consume.[1]

(c) Write down the Bellman equation for the household.

(d) Solve for the equilibrium consumption and investment decision rules as functions of the current state. Hint: What are reasonable guesses given log utility?

(e) Why doesn't expected future policy affect current consumption and investment decisions?

Consider the same economic environment as above, but now allow the government to sell real one-period bonds to finance any discrepancies between spending and revenues. Let b_{t+1} denote bonds sold in period t at price q_t , which pay one unit of consumption goods in period $t + 1$. Agents enter each period t with capital, k_t, and bonds, b_t.

The representative household now chooses

$$\{c_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$$

to maximize utility subject to

$$c_t + k_{t+1} + q_t b_{t+1} = (1 - \tau_t) f(k_t) + b_t$$

k_0 given. Assume the same functional forms before. The household takes as given the time paths of policy variables. The government chooses paths for

$$\{\tau_t, b_{t+1}\}_{t=0}^{\infty}$$

to finance s_t^g according to:

$$q_t b_{t+1} + \tau_t f(k_t) = g_t + b_t$$

b_0 given.

(f) Define the state of the economy and write down Bellman's equation for the household.

(g) Assuming that in a steady state the tax rate and government spending share are given and constant, derive expressions for steady state consumption, capital, and government debt.

[1] You can convince yourself, however, that it would not make a difference if firms were added to the model.

1.1 Answer: Taxes and government spending

(a) It suffices to show that k is bounded, even if c and g both equal zero forever. Since

$$k_{t+1} = k_t^{\theta}$$

the corresponding steady state has $k = 1$. If $k_t > 1$ then $k_{t+1} < k_t$.
(b) The objective function for the household problem is

$$\sum_{t=0}^{\infty} \beta^t u([1-\tau]k_t^{\theta} - k_{t+1})$$

The first-order condition for k is:

$$\frac{\beta^t (1-\tau)\theta k_t^{\theta-1}}{c_t} - \frac{\beta^{t-1}}{c_{t-1}} = 0$$

In steady state this reduces to

$$\beta(1-\tau)\theta = k^{1-\theta}$$

(c) The state is (k_t, s_t^g). Bellman's equation is

$$V(k, s^g) = \max \ln\left((1-s^g)k^{\theta} - k'\right) + \beta V(k', s^{g'})$$

(d) The first-order condition for the control (k') is

$$1/c = \beta V_k(.')$$

The envelope condition is

$$V_k = \left((1-s^g)\theta k^{\theta-1}\right)/c$$

The Euler equation is therefore

$$1/c = (1-s^{g'})\theta k'^{\theta-1}\beta/c'$$

Let

$$y(k, s^g) = (1-s^g)k^{\theta}$$

Guess $c(k, s^g) = \alpha y$ and

$$k'(k, s^g) = (1-\alpha)y$$

Use the Euler equation to solve for α:

$$1/(\alpha y) = \beta\theta(y'/k')/(\alpha y')$$

$\Rightarrow$

$$1-\alpha = \beta\theta$$

Strictly speaking we should now also guess a value function to verify that the policy function together with V satisfies the Bellman equation. Guess:

$$V(k, s^g) = E + F\ln\left((1-s^g)k^{\theta}\right)$$

Therefore,

$$V'(k) = F\theta/k$$

Apply the policy function to the right hand side of the Bellman equation:

$$V(k, s^g) = \ln\left((1-s^g)\alpha k^{\theta}\right) + \beta E + \beta F\left[C_1 + \theta^2\ln(k)\right]$$

Thus,

$$V'(k) = F\theta/k = (1+\beta\theta F)\theta/k$$

Therefore, $F = 1/(1-\beta\theta)$, which is unsurprisingly what we found above for the case without taxes – the tax just serves as a shifter of the production function. The verification step is the same as above.
(e) Future policy does not affect consumption decisions because of log utility.

(f) The state is now (b_t, k_t, s_t^g). The Bellman equation is:

$$V(k, b, s^g) = \max \ln ((1-\tau)f(k) + b - qb' - k') + \beta V(k', b', s^{g'})$$

The FOCs are $\beta V_k(.') = 1/c$, $\beta V_b(.') = q/c$. The envelope conditions are:

$$\begin{aligned} V_k &= (1-\tau)f'(k)/c \\ V_b &= 1/c \end{aligned}$$

Combining them yields:

$$\beta(1-\tau)f'(k')/c' = 1/c$$

and

$$q/c = \beta/c'$$

The first implication is that both assets must yield the same rate of return:

$$1/q = (1-\tau)f'(k')$$

Furthermore, the Euler equation implies that the steady state **interest rate equals the discount rate**:

$$\beta(1-\tau)f'(k) = 1$$

This is a very important finding that arises all the time in infinite horizon models and greatly simplifies the analysis. Therefore $\beta = q$ and

$$k = \{\beta\theta(1-\tau)\}^{1/(1-\theta)}$$

The level of debt follows from the government budget constraint:

$$b = (\tau - s^g)k^\theta/(1-\beta)$$

Finally, the level of consumption can be obtained from the household's budget constraint,

$$c = (1-\tau)f(k) - k + (1-\beta)b$$

2 Leisure in a Growth Model

The economy lasts forever and is populated by a unit measure of identical, infinitely lived households.

(a) Firms maximize period profits, renting capital and labor from households at rental prices r and w, respectively. The technology is constant returns to scale: $Y = F(K, L)$. Derive the first-order conditions for the firm. To fix notation, define $\kappa = K/L$. Be careful to distinguish between capital per person (k) and capital per hour worked (κ)!

(b) Households are endowed with one unit of time in each period that can be used for leisure (l) or work ($n = 1 - l$). Households solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t.

$$c_t + k_{t+1} = w_t n_t + (1 + r_t)k_t$$

$n_t + l_t = 1$, k_0 given.

Capital does not depreciate. The utility function u is strictly increasing in both arguments, strictly concave, and twice continuously differentiable. Formulate the household problem as a dynamic program. Derive conditions characterizing optimal household behavior.

(c) Define a competitive equilibrium.

(d) Derive expressions characterizing the steady state capital stock, labor supply and consumption. This should be a set of 3 equations that contain only (c, k, n) as endogenous variables.

(e) Assume that the production function is Cobb-Douglas,

$$F(K, L) = K^\theta L^{1-\theta}$$

and that preferences are log:

$$u(c,l) = \ln(c) + \gamma \ln(l)$$

Solve for labor supply (n) in closed form. (f) How does the solution for n change, if preferences are instead

$$u(c,l) = \ln(c) + \gamma l$$

(maintaining the Cobb-Douglas technology)?

2.1 Answer: Leisure in a growth model

(a) These are standard:

$$\begin{aligned} r &= f'(\kappa) \\ w &= f(\kappa) - f'(\kappa)\kappa \end{aligned}$$

where $\kappa = K/L$.

(b) The state is simply k. One version of the Bellman equation is

$$V(k) = \max u(wn - (1+r)k - k', 1-n) + \beta V(k')$$

with controls n and k'. The FOCs are

$$\begin{aligned} u_c w &= u_l \\ u_c &= \beta V'(k') \end{aligned}$$

The envelope condition is

$$V'(k) = u_c(1+r)$$

This yields the standard Euler equation:

$$u_c = \beta(1+r')u_c(.')$$

For completeness, the Lagrangian solution:

$$\begin{aligned} \Gamma &= \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right) \\ &\quad + \sum_{t=0}^{\infty} \lambda_t \left[w_t(1-l_t) + (1+r_t)k_t - c_t - k_{t+1}\right] \end{aligned}$$

First-order conditions:

$$\begin{aligned} c_t &: \beta^t u_c\left(t\right) = \lambda_t \\ l_t &: \beta^t u_l\left(t\right) = \lambda_t w_t \\ k_{t+1} &: \lambda_t = \lambda_{t+1}\left(1 + r_{t+1}\right) \end{aligned}$$

A solution to the household problem (in terms of sequences) is a sequence

$$(c_t, l_t, n_t, k_t)$$

that satisfies the static FOC, the Euler equation, $n + l = 1$, and the budget constraint. In sequence notation there should also be a TVC:

$$\lim_{t\to\infty} \beta^t u_c(c_t, l_t)k_t = 0$$

(c) Market clearing conditions are: $K = k$ (or $\kappa = k/n$), $L = n$, and

$$F(k,n) + k = c + k'$$

A CE is then a list of sequences ($c,\ n,\ l,\ k,\ w,\ r,\ K,\ L$) that satisfies 4 household conditions, 2 firm conditions, 3 market clearing conditions.

(d) From the Euler equation:

$$f'(\kappa) = r = 1/\beta - 1$$

which solves for κ.

Goods market clearing then implies $c/n = f(\kappa)$. The static optimality condition provides the 3^{rd} equation: $w = u_l/u_c$, where w is determined by the firm's FOC.

(e) The 3 conditions from (d) specialize to:

$$\begin{aligned} 1/\beta - 1 &= \theta\kappa^{\theta-1} \\ c/n &= \kappa^{\theta} \\ (1-\theta)\kappa^{\theta} &= \gamma c/(1-n) \end{aligned}$$

Therefore

$$\kappa = \left(\frac{\theta}{1/\beta - 1}\right)^{1/(1-\theta)}$$

$$\frac{c}{n} = \kappa^{\theta} = \frac{1-n}{n\gamma}(1-\theta)\kappa^{\theta}$$

or

$$\frac{1-n}{n} = \frac{\gamma}{1-\theta}$$

This makes sense: n is between 0 and 1. If the household doesn't care about leisure ($\gamma = 0$), then $n = 1$. If the household really loves leisure ($\gamma \to \infty$), then $n \to 0$.

(f) Now the 3 steady state conditions are

$$\begin{aligned} 1/\beta - 1 &= \theta\kappa^{\theta-1} \\ c/n &= \kappa^{\theta} \\ (1-\theta)\kappa^{\theta} &= \gamma c \end{aligned}$$

Therefore

$$c = n\kappa^{\theta} = (1-\theta)\kappa^{\theta}/\gamma$$

$\Rightarrow$

$$n = \frac{1-\theta}{\gamma}$$

3 Non-separable Utility

Consider the following growth economy, modified to include (i) costs to adjusting the capital stock and (ii) habit persistence in consumption.

The social planner solves

$$\max \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$

subject to the feasibility constraints

$$\begin{aligned} c_t + x_t &= f(k_t) \\ k_{t+1} &= x_t + (1-\delta)k_t - g(x_t/k_t) \end{aligned}$$

f satisfies Inada conditions. g is a strictly increasing and convex function. Compute and interpret the first-order necessary conditions for the planner's problem.

3.1 Answer: Non-separable utility

Lagrangian

$$
\begin{aligned}
\Gamma &= \sum\nolimits_{t=1}^{\infty} \beta^t u\left(f(k_t) - x_t, f(k_{t-1}) - x_{t-1}\right) \\
&\quad + \sum\nolimits_{t=1}^{\infty} \lambda_t \left(x_t - g(x_t/k_t) + (1-\delta)k_t - k_{t+1}\right)
\end{aligned}
$$

First order conditions:

$$
\begin{aligned}
& \beta^t u_1(t, t-1) + \beta^{t+1} u_2(t+1, t) \\
=\ & \lambda_t \left(1 - \partial g/\partial x_t\right) \\
& f'(k_t)\left(\beta^t u_1(t, t-1) + \beta^{t+1} u_2(t+1, t)\right) \\
=\ & \lambda_t \left(1 - \delta - \partial g/\partial k_t\right) - \lambda_{t-1}
\end{aligned}
$$

Euler equation:

$$
\lambda_{t-1} = \lambda_t \left[1 - \delta - \partial g/\partial k_t + f'(k_t)\{1 - \partial g/\partial x_t\}\right]
$$

Define the total marginal utility of consumption as

$$
U'(c_{t-1}) = \beta^{t-1} u_1(t-1, t-2) + \beta^t u_2(t, t-1)
$$

The Euler Equation then becomes:

$$
\begin{aligned}
U'(c_{t-1}) &= [1 - \partial g/\partial x_{t-1}] U'(c_t) \times \\
&\quad \left(f'(k_t) + \frac{1 - \delta - \partial g/\partial k_t}{1 - \partial g/\partial x_t}\right)
\end{aligned}
$$

3.1.1 Interpretation

$$
\begin{aligned}
U'(c_{t-1}) &= [1 - \partial g/\partial x_{t-1}] U'(c_t) \times \\
&\quad \left(f'(k_t) + \frac{1 - \delta - \partial g/\partial k_t}{1 - \partial g/\partial x_t}\right)
\end{aligned}
$$

Give up one unit of c_{t-1}. This costs $U'(c_{t-1})$.
We can increase x_{t-1} by 1 and raise k_t by $[1 - \partial g/\partial x_{t-1}]$
We eat the results next period at marginal utility $U'(c_t)$.
We can eat

- the additional output $f'(k_t)$;
- the undepreciated capital $1 - \delta$;
- the reduction in the adjustment cost $-\partial g/\partial k_t > 0$;
- the reduced adjustment cost due to lower investment (the $\partial g/\partial x$ in the denominator).

A **solution** of the hh problem is:
Sequences $\{x_t, k_t\}$ that satisfy

1. the Euler equation
2. the flow budget constraint.
3. The boundary conditions k_1 given and $k_t \geq 0$.

4 Brock-Mirman Model

Consider a Robinson-Crusoe economy where the consumer solves

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$\begin{aligned} k_{t+1} &= Ak_t^\theta - c_t \\ & \quad k_0 \text{ given} \\ c_t &\geq 0, k_t \geq 0 \end{aligned}$$

(a) Show that any feasible path (c_t, k_t) is bounded.

(b) Explain why neither zero consumption nor zero saving occurs along the optimal path.

(c) Write down Bellman's equation and derive the Euler equation

$$c_{t+1}/c_t = \theta\beta A k_{t+1}^{\theta-1}$$

(d) The value function is $V(k) = E + F\ln(k)$, where

$$F = \theta/(1-\theta\beta)$$

and

$$E = [\ln(A[1-\theta\beta]) + \ln(A\theta\beta)\theta\beta/(1-\theta\beta)]/(1-\beta)$$

Prove this result using the method of undetermined coefficients. That is, plug the expression for V into Bellman's equation and compare terms on both sides.

4.1 Answer. Brock-Mirman Model

(a) The lower bounds are obviously zero. An upper bound for k can be derived by assuming that the household invests all output forever. Since the marginal product of k goes to zero as $k \to \infty$, the capital stock converges to the level where

$$\bar{k} = A\bar{k}^\theta$$

(draw a graph to cinvince yourself). Then consumption is obviously bounded by the maximum output.

(b) Zero consumption would imply infinite marginal utility and negative infinite utility level. Zero investment would imply zero consumption next period.

(c) Bellman's equation is

$$V(k) = \max \ln(Ak^\theta - k') + \beta V(k')$$

The FOC is

$$u'(c) = \beta V'(k')$$

The envelope condition is

$$V'(k) = u'(c)A\theta k^{\theta-1}$$

This implies the Euler equation

$$u'(c) = \beta u'(c')A\theta k^{\theta-1}.$$

With log utility this implies the expression given in the question.

(d) With the guess for V Bellman's equation becomes

$$V(k) = \max \ln(Ak^\theta - k') + \beta[E + F\ln(k')]$$

The FOC is

$$1/(Ak^\theta - k') = \beta F/k'$$

or

$$k' = \beta F(Ak^\theta - k')$$

Solving for k'

$$k' = Ak^{\theta}\beta F/(1+\beta F)$$

allows us to eliminate k' from the value function

$$\begin{aligned} V(k) &= \ln\left(Ak^{\theta}(1-\beta F/[1+\beta F])\right) + \beta\left\{E + F\ln\left(Ak^{\theta}\beta F/(1+\beta F)\right)\right\} \\ &= \ln\left(Ak^{\theta}\right) - \ln(1+\beta F) + \beta E + \beta F\left\{\ln(Ak^{\theta}) + \ln\left(\beta F/(1+\beta F)\right)\right\} \end{aligned}$$

Collecting terms we need

$$F = \theta + \beta F\theta$$

and

$$E = \ln(A) - \ln(1+\beta F) + \beta E + \beta F\left\{\ln(A) + \ln\left(\beta F/(1+\beta F)\right)\right\}$$

The first equation yields F. The second one (after substituting in F) does produce the right E, although it may not look like it.

5 Endogenous Discount Factor

Consider the following version of the neoclassical growth model with endogenous discounting. The representative household solves

$$\max \sum_{t=1}^{\infty}\left[\prod_{n=0}^{t-1}\beta(a_n)\right]\ln c_t$$

subject to

$$k_{t+1} + a_t + c_t = A\,k_t^{\alpha} + (1-\delta)\,k_t$$

where k_1 and a_0 are given and the parameters satisfy $A > 0$, $0 < \alpha \leq 1$, $0 < \delta < 1$.

The function $\beta(a)$ is continuous, increasing, differentiable, and strictly concave with $0 < \beta(a) < 1$ for all a. Note that this differs from the standard growth model only in that the negative effects of discounting can be reduced by paying a cost a_t. Note further that today's expenditure a_t affects how consumption from $t+1$ onwards is discounted.

(a) Derive and explain the household first-order conditions.

(b) Assume that $\alpha < 1$. Characterize the stationary optimal solution. Explain whether it is unique or not.

(c) Assume that $\alpha = 1$. Explain why this economy may exhibit negative long-run growth. Provide sufficient conditions for positive growth.

(d) Is it possible for this economy to exhibit positive long-run growth when $\alpha < 1$? Explain.

5.1 Answer Sketch: Endogenous Discount Factor

(a) Bellman equation:

$$V(k) = \max\, u(c) + \beta(a)\,V\left(A\,k^{\alpha} + [1-\delta]\,k - c - a\right)$$

First-order conditions:

$$\begin{aligned} u'(c) &= \beta(a)\,V'(k') \\ \beta'(a)\,V(k') &= \beta(a)\,V'(k') \\ &= u'(c) \end{aligned}$$

Envelope condition:

$$V'(k) = \beta(a)\,V'(k')\,R(k)$$

where $R(k) = \left[\alpha\,A\,k^{\alpha-1} + 1 - \delta\right]$ is the marginal product of capital. Euler equation:

$$u'(c) = \beta(a')\,u'(c')\,R(k')$$

(b) Stationary solution is characterized by a pair of equations in (k, a):

$$\begin{aligned}
\beta(a) &= 1/R(k) \\
\frac{u'\left(f(k) - \delta\, k - a\right)}{\beta'(a)} &= \sum_{t=1}^{\infty} \beta(a)^t \ln\left(f(k) - \delta\, k - a\right)
\end{aligned}$$

There is no reason why the solution to this pair of equations should be unique. Households may choose high c and low a or vice versa. Of course, the question whether the optimal trajectory converges to a unique long-run stationary solution is harder.

(c) Simple answer: Balanced growth rate of consumption is $1 + g(c) = \beta(a)\, R(k) = \beta(a)\, [A - \delta]$. If $\beta(a) < [A - \delta]^{-1}$ there cannot be positive growth. Conversely, if $\beta(a) > [A - \delta]^{-1}$ for all a there will be positive growth.

(d) No. Obvious.

6 Utility from government spending

Consider a representative agent one sector growth model with preferences given by

$$\sum_{t=0}^{\infty} \beta^t\, u(c_t, g_t),$$

where $u(c, g) = \left[c^{\alpha}\, g^{1-\alpha}\right]^{1-\sigma} / (1 - \sigma)$ with $0 < \alpha < 1$ and $\sigma > 0$. The flow of goods provided by the government, g_t, is taken as given by the household. The aggregate resource constraints are given by

$$c_t + \gamma_t\, g_t + x_t = f(k_t) \tag{1}$$

$$k_{t+1} = (1 - \delta)\, k_t + x_t. \tag{2}$$

Assume that $f(.)$ satisfies any regularity conditions you need. The government uses lump sum taxes τ_t or income taxes μ_t to finance spending. γ_t is the relative cost of government goods. The government budget is balanced in each period.

(a) Define a competitive equilibrium.

(b) Assume that the income tax rate is zero, that $\gamma_t = \gamma$ and that the government chooses g_t to maximize the utility of the representative agent. Derive the equilibrium ratio g_t/c_t.

Show the following two observations: (i) The steady state interest rate and (ii) the ratio $c/f(k)$ are both independent of γ. Explain the intuition underlying both findings. *Hint:* It may be easier to solve a planning problem rather than a competitive equilibrium.

(c) Now assume that the lump sum tax rate is zero and that $\gamma_t = \gamma$. Government spending is not chosen optimally, but fixed exogenously. The income tax rate balances the government budget. Determine whether an infinitesimal change in γ increases or decreases steady state output (your answer will depend on the initial level of γ). Why does output now depend on γ, while it did not in part (b)?

6.1 Answer: Utility from government spending

(a) Households solve the following problem.

$$\max \sum_{t=0}^{\infty} \beta^t\, u(c_t, g_t)$$

subject to

$$c_t + x_t + \tau_t = w_t + r_t k_t$$

$$k_{t+1} = (1 - \delta)\, k_t + x_t.$$

$$\lim_{t\to\infty} \beta^t u_c(c_t, g_t)\, k_t = 0$$

Firms solve

$$\max F(k_t, n_t) - w_t^G n_t - r_t^G k_t$$

with the usual first-order conditions $w_t^G = f(k_t) - f'(k_t)k_t$ and $r_t^G = f'(k_t)$. Here I (prematurely) impose the equilibrium condition $n_t = 1$.

The government budget constraint is $\tau_t + \mu_t f(k_t) = \gamma_t g_t$.

A competitive equilibrium is a set of sequences $(c_t, k_t, n_t, g_t, \tau_t, \mu_t, x_t, w_t, r_t, w_t^G, r_t^G)$ such that:

- (c_t, k_t, x_t) solve the household problem;
- (w_t^G, r_t^G) solve the firm problem.
- All but one of the policy variables (g_t, τ_t, μ_t) must be exogenous with the last one satisfying the government budget constraint.
- The price identities $w_t = (1 - \mu_t)\, w_t^G$ and $r_t = (1 - \mu_t)\, r_t^G$ hold.
- The labor market clears: $n_t = 1$. The capital market clears (built into the notation).
- The goods market clears, i.e., (1) and (2) hold.

We have 11 objects and 12 equations, one being redundant by Walras law.

(b) It is easier to solve the planner's problem than the competitive equilibrium problem. The planner solves

$$\max \sum_{t=0}^{\infty} \beta^t\, u(c_t, g_t)$$

subject to feasibility constraints (1) and (2), which are best combined into

$$k_{t+1} = f(k_t) + (1 - \delta)\, k_t - c_t - \gamma_t g_t. \tag{3}$$

First order conditions are

$$u_c(t) = \lambda_t$$

$$u_g(t) = \gamma_t \lambda_t$$

$$\lambda_t = \beta\, \lambda_{t+1}\, [1 - \delta + f'(k_{t+1})] \tag{4}$$

Given the utility function, this simplifies to

$$u_c(t) = \frac{u(t)}{c_t}\, \alpha\, (1 - \sigma)$$

$$u_g(t) = \frac{u(t)}{g_t} (1 - \alpha)\, (1 - \sigma)$$

It follows that

$$\frac{g_t}{c_t} = \frac{1 - \alpha}{\alpha\, \gamma_t} \tag{5}$$

Note that a higher γ implies a lower g/c. The steady state interest rate is determined by the Euler equation (4): $1/\beta = 1 - \delta + f'(k^*)$, which shows observation (i): the steady state interest rate is independent of γ. The intuition is that the long-run supply of capital is infinitely elastic in this kind of model. The lump-sum tax has a wealth effect on consumption, but does not affect long-run capital.

The consumption-output ratio is obtained from the feasibility condition (3) together with (5): $c^* = \alpha\,[f(k^*) - \delta\,k^*]$. Consistent with observation (ii) this does not depend on γ. This is due to the logarithmic period utility function. It is optimal to spend fraction α of current consumption on good c, regardless of its relative price γ.

(c) The easiest approach here is to set up another planning problem. In addition to feasibility, we impose a constraint indicating that tax revenues equal government spending: $\mu_t\,f(k_t) = \gamma\,g_t$. This should be thought of as a function $\mu(k;\gamma)$. In addition, we impose that resources for consumption and investment are given by $k_{t+1} = (1-\mu_t)\,f(k_t) + (1-\delta)\,k_t - c_t$. When taking first-order conditions with respect to k it is important to keep in mind that the private sector takes μ as given. It should be apparent that the Euler equation now requires $\lambda_t = \beta\,\lambda_{t+1}\,[1-\delta+(1-\mu_t)\,f'(k_{t+1})]$ so that the steady state capital stock is determined by

$$1/\beta + \delta - 1 = (1-\mu)\,f'(k) = (1-\gamma\,g/f(k))\,f'(k) \tag{6}$$

The effect of changing γ is found by applying the implicit function theorem to (6) which yields

$$\frac{\partial k}{\partial \gamma} = -\frac{f'(k)\,g}{f''(k)\,[f(k)-\gamma g] + f'(k)^2\,\gamma g/f(k)}$$

For small initial γg a higher γ implies a higher k, but for γg close to $f(k)$ (it obviously cannot be any larger), the derivative is negative. The intuition is related to the idea that the distortion associated with a tax increases with the square of the tax rate. For small γg the initial tax rate is close to zero and a small increase in tax revenues is not very detrimental for capital accumulation.

7 Human and Physical Capital

Consider an economy with two types of capital (k,h) and with land (L). Output is produced according to the constant returns to scale production function $F(k,h,L)$. F obeys Inada conditions. The feasibility constraints are

$$\begin{aligned} F(k,h,L) &= c + x_k + x_h \\ k' &= (1-\delta)\,k + x_k \\ h' &= (1-\delta)\,h + x_h \end{aligned}$$

Land is in fixed supply. Output is produced by firms who rent factor inputs from households. There is a single representative household who maximizes discounted utility $\sum \beta^t\,u(c_t)$.

Markets: There are rental markets for k and h with rental prices q_k and q_h. There is a rental market for land with price q_L and a resale market for land with price p_L. Think of h as human capital and of $q_h\,h$ as labor earnings.

(a) State the household problem as a Dynamic Program. Define a solution in sequence language.

(b) Define a competitive equilibrium.

(c) Assume that $F(k,h,L) = A\,k^{\alpha}\,h^{\varphi}\,L^{1-\alpha-\varphi}$. Characterize the equilibrium path of the wage-rental ratio q_{ht}/q_{kt} and the capital-labor ratio k_t/h_t. Note that there is no non-negativity constrain on investment (x_{ht}, x_{kt}).

(d) Consider two economies that differ only in scale. That is, economy B is endowed with a fraction b of economy A's k and h. Assume that both economies are below their steady state levels of k and h (so that investment is positive). In which country do we see higher wage and rental rates (q_h, q_k). Should migrants move from poor to rich countries or the other way around?

7.1 Answer: Human and Physical Capital

Based on a question due to Rody Manuelli.

(a) Bellman equation

$$\begin{aligned} V(k,h,L) = {} & \max\, u(c) + \beta\,V\left([1-\delta]\,k + x_k, [1-\delta]\,h + x_h, L'\right) - \\ & \lambda\,\{c + x_k + x_h + p_L\,(L'-L) - q_h h - q_k k - q_L L\} \end{aligned}$$

FOCs

$$
\begin{aligned}
u'(c) &= \lambda \\
\beta V_L(.') &= \lambda p_L \\
\beta V_k(.') &= \lambda \\
\beta V_h(.') &= \lambda
\end{aligned}
$$

Envelope conditions

$$
\begin{aligned}
V_k &= \beta V_k(.') [1-\delta] + \lambda q_k \\
V_h &= \beta V_h(.') [1-\delta] + \lambda q_h \\
V_L &= \lambda [p_L + q_L]
\end{aligned}
$$

Therefore, $V_k(.') = V_h(.') = V_L(.')/p_L$. This implies the no arbitrage conditions

$$1 - \delta + q'_k = 1 - \delta + q'_h = [p'_L + q'_L]/p_L$$

In words: all three assets must have the same rate of return. Denote this by $R = 1 - \delta + q_k$.

Euler:

$$u'(c) = \beta u'(c') R'$$

Solution: $\{c_t, x_{kt}, x_{ht}, k_t, h_t, L_t\}$ that satisfy: Euler equation, budget constraint, 2 laws of motion, 2 arbitrage conditions.

TVC: $\lim \beta^t u'(c_t) k_t = 0$, $\lim \beta^t u'(c_t) h_t = 0$, $\lim \beta^t u'(c_t) L_t = 0$

(b) Competitive Equilibrium:

Firms have standard FOCs: $F_x = q_x$ for $x = k, h, L$.

Sequences $\{c_t, x_{kt}, x_{ht}, k_t, h_t, L_t, q_{kt}, q_{ht}, q_{Lt}, p_{Lt}\}$ that satisfy:

6 household conditions

3 firm conditions

Factor market clearing is implicit in notation. Counts as one equation (L_t given). Goods market clearing is the same as feasibility.

(c) We already know from the household problem that the wage rental ratio is 1 at all dates. We can then solve for $k/h = \alpha/\varphi$ from the firm's first-order conditions. If we had non-negativity constraints on investment, this would only hold at dates with positive investment.

(d) We can solve for the wage rate using the known k/h from (c). We find that richer countries have lower wage rates (per efficiency unit). Migration should flow from rich to poor countries. Which suggests that something else must be going on in the data (rich countries have high tfp).

8 Heterogeneity

9 Productive Government Capital

Consider a standard growth model with a single extension: the government imposes lump-sum taxes on the household in order to buy productive government capital $\left(K^G\right)$.

(a) There is a single representative firm which maximizes profits taking rental prices for labor (w_t) and capital (q_t) as given. The technology is

$$F(K_t, L_t; K_t^G) + (1-\delta)(K_t + K_t^G) = K_{t+1} + K_{t+1}^G + C_t.$$

The notation is as usual: K is capital, L is labor. The firm takes K^G as given and do not pay for it (think infrastructure). F has constant returns to scale in all three inputs. Define a solution to the firm's problem.

(b) There is a single representative household who maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to a budget constraint. The household inelastically supplies one unit of labor and rents its capital to firms. It pays a lump-sum tax τ_t to the government. Capital is the only asset. Define a solution to the household's problem using both Dynamic Programming and sequence methods.
(c) Define a competitive equilibrium using sequence notation. Assume that the government imposes an exogenous tax τ in each period.
(d) Characterize the solution to the problem facing the central planner.
(e) Assume that the technology is of the form

$$F(K, L; K^G) = K^{\alpha}(K^G)^{\varphi} L^{1-\alpha-\varphi}$$

Determine the steady state capital stock, given that the government imposes the welfare maximizing tax rate τ. *Hint:* Here you will have to use what you found for the planning problem. This answer is short.

Answer: Productive Government Capital

(a) This is standard: $q = F_K; w = F_L$. It is not correct to write $q = f'(k)$ because from the point of view of the firm F has diminishing returns to scale.
(b) The household solves

$$V(k) = \max u((1 - \delta + r)k + w + \pi - \tau - k') + \beta V(k')$$

The FOC is

$$u'(c) = \beta V'(k')$$

The envelope condition is

$$V'(k) = u'(c)(1 - \delta + r)$$

The Euler equation is standard:

$$u'(c) = \beta(1 - \delta + r')u'(c')$$

A solution in sequence notation is a sequence of (c_t, k_t) which satisfies the Euler equation, the budget constraint, and a transversality condition,

$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0.$$

In functional language: A solution is a value function V and a policy function $k' = h(k)$ that satisfy:

- V is a fixed point of the Bellman equation given the policy function h:

$$V(k) = u((1 - \delta + r)k + w + \pi - \tau - h(k)) + \beta V(h(k))$$

- The policy function solves the max problem:

$$h(k) = argmax\, u((1 - \delta + r)k + w + \pi - \tau - k') + \beta V(k')$$

(c) A CE is a sequence of quantities $\left(c, k, K, L, K^G, \pi\right)$ and prices (q, w, r) (8 variables) that satisfy:

- 2 household optimality conditions;
- 2 firm FOCs;
- the government budget constraint:

$$K^G_{t+1} = (1 - \delta)K^G_t + \tau$$

- market clearing: $L_t = 1$; $k_t = K_t$; goods market clearing (same as feasibility); the identity $q = r$, and the definition of π.

(d) The planner maximizes utility subject to the feasibility constraint only (and $L = 1$). Optimality requires the static condition $F_K = F_{K^G}$ and the Euler equation

$$u'(c) = \beta(F_K(.') + 1 - \delta)u'(c').$$

(e) We need the Euler equation and the planner's static condition. $F_K = F_{K^G}$ implies

$$K^G/K = \varphi/\alpha$$

Then

$$F_K = \alpha(\varphi/\alpha)^{\varphi} K^{\alpha+\varphi-1} = 1/\beta - (1-\delta)$$