AK Model:
Phase Diagram

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Introduction

We study an endogenous growth model with transitional dynamics. The model is asymptotically AK.
As an example of a phase diagram with endogenous growth.
We modify the Ak model’s production function:

\[ H(K,L) = AK + F(K,L) \]  \hspace{1cm} (1)

In intensive form

\[ h(k) = Ak + f(k) \]

where \( F(K,L) = Lf(k) \) satisfies Inada conditions and has constant returns to scale in \( K \) and \( L \) jointly.

For simplicity, assume \( f(k) = k^\alpha \) with \( \alpha < 1 \).
The only change to the equilibrium conditions of the $Ak$ model: the marginal product of capital is not $A$ but

$$H_K(K, L) = A + F_K(K, L) = A + f'(k)$$  \hspace{1cm} (2)

Laws of motion:

$$\dot{k} = h(k) - (n + \delta)k - c$$  \hspace{1cm} (3)

$$g(c) = (h'(k) - \delta - \rho)/\sigma$$  \hspace{1cm} (4)

Asymptotically, $f'(k) \rightarrow 0$ and the model becomes $Ak$. 

Equilibrium
Phase Diagram with Endogenous Growth

How to draw a phase diagram when $c$ and $k$ grow at endogenous rates?
One approach: Find ratios that are constant asymptotically
For inspiration, start from

$$g(k) = \frac{h(k)}{k} - (n + \delta) - \frac{c}{k}$$

(5)

That suggests to try:

- $z = \frac{h(k)}{k}$
- $x = \frac{c}{k}$.

Another approach: **detrend** the model and then draw the phase diagram.
Laws of motion

\[ g(z) = g(h(k)) - g(k) \quad (6) \]
\[ g(x) = g(c) - g(k) \quad (7) \]

- We therefore need to find expressions for \( g(h(k)) \), \( g(k) \), and \( g(c) \) in terms of \( z \) and \( x \) only.

- First rewrite the law of motion for \( k \) as

\[ g(k) = \frac{h(k)}{k} - \delta - n - \frac{c}{k} \quad (8) \]
\[ = z - \delta - n - x \quad (9) \]
Laws of motion

Next, \( g(c) = [h'(k) - \delta - \rho] / \sigma \).

We need to replace \( h'(k) \).

Note that

\[
h'(k) = A + \alpha f(k)/k = A + \alpha (z - A) = \alpha z + (1 - \alpha)A
\]

Use this to rewrite (4) as

\[
g(c) = \frac{\alpha z + (1 - \alpha)A - \delta - \rho}{\sigma}
\]
Finally,

\[ g(h(k)) = \frac{h'(k)k}{h(k)} g(k) = \frac{\alpha z + (1 - \alpha)A}{z} g(k) \]
Laws of motion

\[ g(z) = g(h(k)) - g(k) \]
\[ = \left[ \frac{\alpha z + (1 - \alpha)A}{z} - 1 \right] [z - x - n - \delta] \]
\[ = (1 - \alpha) \left( \frac{A}{z} - 1 \right) [z - x - n - \delta] \]

and

\[ g(x) = g(c) - g(k) \]
\[ = \frac{\alpha z + (1 - \alpha)A - \rho - \delta}{\sigma} - z + x + n + \delta \]
\[ = \varphi + x + z(\alpha/\sigma - 1) \]

where \( \varphi = n + \delta + (1 - \alpha)A/\sigma - (\rho + \delta)/\sigma \).
\[ \dot{x} = 0 \text{ requires} \]

\[ x_{ss} = \left(1 - \frac{\alpha}{\sigma}\right)z_{ss} - \varphi \quad (10) \]

For realistic parameter values (e.g. \( \alpha \approx 0.3 \) and \( \sigma \geq 1 \)), we have

\[ 0 < 1 - \frac{\alpha}{\sigma} < 1. \]

- Negative intercept.
- Slope \(< 1\).
\( \dot{z} = 0 \) Locus

\( \dot{z} = 0 \) has two solutions:

- \( z = A \) or
- \( x = z - n - \delta \).

In steady state:

\[ z_{ss} = A \quad \text{(11)} \]

because

\[ \lim_{k \to \infty} z = \lim_{k \to \infty} \frac{Ak + f(k)}{k} = A \quad \text{(12)} \]
\[ \dot{z} = 0 \] Locus

But for finite \( k \): \( z = A/k + f(k)/k > A \).

Therefore the relevant condition is

\[ x = z - n - \delta \] (13)

- Negative intercept
- Slope = 1
Summary

Laws of motion:

\[
\begin{align*}
\dot{z} & = (1 - \alpha)(A - z)(z - x - n - \delta) \\
g(x) & = \varphi + x + z(\alpha/\sigma - 1)
\end{align*}
\]

\[
\begin{align*}
\dot{z} & = 0 : x = z - n - \delta \\
\dot{x} & = 0 : x = -\varphi + (1 - \alpha/\sigma)z
\end{align*}
\]

Steady state: \( z = A \).

Otherwise: \( z > A \)
Phase Diagram with Endogenous Growth
This system is **saddle-path stable**.

Consider \( x_0 \) above the saddle path

- \( x = c/k \) grows over time; in fact \( g(x) > 0 \) (strictly above \( \dot{x} = 0 \)) implies \( x \to \infty \)
- \( z = h(k)/k \) grows over time; that means that \( k \) must shrink
- feasibility requires \( g(k) = h(k)/k - \delta - c/k \); therefore \( k \to 0 \)
- but then \( g(c) \to \infty \) which violates TVC and feasibility
Stability

Consider $x_0$ below the saddle path

- if above $\dot{x} = 0$: it eventually crosses above the saddle, which is impossible
- otherwise $g(z) < 0$ until $z \to A$ which implies $k \to \infty$
- then $g(u') \to A - \delta - \rho$
- this violates TVC: $\lim_{t \to \infty} e^{-\rho t} u'(c_t) k_t \to \infty$

Both $x$ and $z$ converge monotonically to the steady state.
The important point is the general approach for dealing with the dynamics of growing economies:

1. Write out the equilibrium conditions as usual.
2. Find conditions characterizing the balanced growth path.
3. Find ratios that are constant on the balanced growth path ($x$ and $z$).
4. Express the laws of motion of the economy in terms of these ratios.

An alternative approach is to transform the economy into stationary form before characterizing its equilibrium.
Reading

- Acemoglu (2009), ch. 11.