# Review Problems: Innovation and Growth

Econ720. Fall 2017. Prof. Lutz Hendricks

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## 1 Technology Adoption

[Based on Romer (2001) question 3.12] The world consists of 2 regions: $i = N, S$. Output is produced according to

$$Y_i = K_i^{\alpha} \; (A_i \; [1 - a_{Li}] \, L_i)^{1-\alpha}$$

Capital is accumulated according to

$$\dot{K}_i = s_i \; Y_i$$

New technologies are developed in the North:

$$\dot{A}_N = B \; a_{LN} \; L_N \; A_N.$$

The South learns from the North:

$$\dot{A}_S = \mu \; a_{LS} \; L_S \; (A_N - A_S)$$

as long as $A_N > A_S$. Otherwise $\dot{A}_S = 0$.

Labor supplies ($L_i$) are constant over time.

1. Determine the long-run growth rate of $Y_N$.

2. Define $Z = A_S/A_N$. Derive the law of motion for $Z$. Is $Z$ stable? If so, which value does it converge to? What is the long-run growth rate of $Y_S$?

3. Assume $a_{LN} = a_{LS}$ and $s_N = s_S$. Find the balanced growth output ratio $Y_N/Y_S$.

## 1.1 Answer: Technology adoption

**1.** The growth rates obey

$$\begin{aligned} g(Y_i) &= \alpha g(K_i) + (1-\alpha)\ g(A_i) & (1) \\ g(K_i) &= s_i Y_i / K_i & (2) \\ g(A_N) &= B\ a_{LN}\ L_N & (3) \\ g(A_S) &= \mu\ a_{LS}\ L_S\ \ (A_N/A_S - 1) & (4) \end{aligned}$$

In the long-run

$$g(Y_i) = g(K_i)$$

and therefore

$$g(Y_i) = g(A_i) \tag{5}$$

**2.** The law of motion for $Z$ is

$$\begin{aligned} g(Z) &= g(A_S) - g(A_N) \\ &= \mu\ a_{LS}\ L_S\ \ (1/Z - 1) - B\ a_{LN}\ L_N \end{aligned}$$

This has a unique, stable steady state. Its value is

$$1/Z^* = 1 + \frac{B\ a_{LN}\ L_N}{\mu\ a_{LS}\ L_S}$$

This implies that larger adopting countries have higher productivity $(Z)$ - the scale effect in action.

**3.** Equals $s_i$ implies equal $K/Y$. Write the production function as

$$Y_i/L_i = (K_i/Y_i)^{\alpha/(1-\alpha)}\ \ A_i\ \ (1 - a_{Li}) \tag{6}$$

Then the ratio of output per worker is

$$y_S/y_N = Z\ \frac{1 - a_{LS}}{1 - a_{LN}} \tag{7}$$

so that larger adopting countries are richer.

## 2 Delayed Adoption

[Based on Romer (2001) question 3.13] The world consists of two regions: $i = N, S$. The North is described by

$$\begin{aligned} Y_N &= A_N \ (1 - a_{LN}) \ L_N \\ \dot{A}_N &= a_L \ L_N \ A_N \end{aligned}$$

The South uses technologies developed in the North with a lag of $\tau$:

$$\begin{aligned} A_S(t) &= A_N(t - \tau) \\ Y_S(t) &= A_S(t) \ L_S \end{aligned}$$

Labor inputs are constant over time.

If $g(Y) = 0.03$, how old must technologies be in the South to yield a 10-fold output gap? Assume $a_L$ is close to 0 and $n = 0.01$.

### 2.1 Answer: Delayed adoption

From the production functions

$$\frac{y_N(t)}{y_S(t)} = \frac{(1 - a_{LN}) A_N(t)}{A_N(t - \tau)}$$

We next find the ratio $A_N(t)/A_N(t-\tau)$ which measures how far behind the South is in terms of technology. $A$ grows at rate

$$g(A) = g(Y) - n$$

so that

$$A_N(t)/A_N(t-\tau) = e^{[g(Y)-n]\tau}.$$

If $a_{LN} = 0$ we need the $A$ gap to equal 10:

$$10 = e^{0.02\tau}$$

or

$$\begin{aligned} \tau &= \ln(10)/0.02 \\ &\simeq 115 \end{aligned}$$

In words: the technology gap would have to be 115 years. It follows that delayed adoption by itself cannot account for large cross-country income gaps. It must also be the case that technologies are used inefficiently.

# 3 Ideas produced from capital and labor

Consider the following version of a Romer model. The Solow block:

$$Y = K_Y^{\alpha} L_Y^{1-\alpha} \tag{8}$$
$$\dot{K} = A s_K Y - \delta K \tag{9}$$
$$L_t = L_0 e^{nt} \tag{10}$$

Production of ideas:

$$\dot{A} = \delta K_A^{\beta} L_A^{\lambda} A^{\phi} \tag{11}$$
$$0 < \phi < 1 \tag{12}$$

Constant behavior:

$$L_Y = s_Y L \tag{13}$$
$$L_A = s_A L \tag{14}$$
$$K_A = a K \tag{15}$$
$$K_Y = (1-a) K \tag{16}$$

1. Find the balanced growth rates of $A, K, Y$.
2. Derive expressions that give the *growth rates* of $g(A)$ and $g(K/L)$ as functions of $g(A)$, $g(K/L)$, and parameters. Set $\delta = 0$.
3. Graph curves representing constant $g(A)$ and $g(K/L)$. Place $g(A)$ on the x-axis and $g(K/L)$ on the y-axis.
4. Which assumptions on the parameters do you need for a balanced growth path to exist (the two curves to intersect)?
5. Is the balanced growth path stable in the case where it exists?

## 3.1 Answer: Ideas produced from capital and labor

**1.** Start from

$$\begin{aligned}(1-\phi) g(A) &= \lambda n + \beta g(K/L) + \beta n \\ g(A) &= (1-\alpha) g(K/L) \\ g(Y/L) &= g(K/L)\end{aligned}$$

These are three equations which can be solved for the 3 growth rates. Some algebra yields

$$g(A) = n\frac{\lambda + \beta}{1 - \phi - \beta/(1-\alpha)} \tag{17}$$

**2.**

$$\begin{aligned} g(g(K)) &= g(A) - (1-\alpha)\, g(K/L) \\ g(g(A)) &= (\beta + \lambda)\, n + \beta g(K/L) - (1-\phi)\, g(A) \end{aligned}$$

**3.-4.** These are straight lines with slopes $(1-\phi)/\beta$ and $1/(1-\alpha)$. For these to intersect, we need the denominator in (17) to be positive.

**5.** Draw the phase diagram to see that the BGP is stable.

# 4 Learning by doing

Consider the following model. Output ($Y$) is produced from capital ($K$) and labor ($L$):

$$Y_t = K_t^{\alpha}\ (A_t L_t)^{1-\alpha} \tag{18}$$

with $0 < \alpha < 1$. Labor grows at the constant rate $n$:

$$L_t = e^{nt} \tag{19}$$

Capital is accumulated according to

$$\dot{K}_t = sY_t. \tag{20}$$

The saving rate $s$ is constant. Knowledge is accumulated as a by-product of production:

$$\dot{A}_t = BY_t^{\varphi}$$

with $0 < \varphi < 1$ and $B > 0$.

1. Derive the *balanced* growth rates of $Y, K$, and $A$.

2. Does a change in $s$ affect long-run growth? Explain the intuition. (You can do this even if you did not answer 1.)

3. Does a change in $n$ affect long-run growth? Explain the intuition. (You can do this even if you did not answer 1.)

4. Derive an expression for the balanced growth *level* of $Y(t)$ as a function of $L(t)$ and parameters. (If you could not solve for the balanced growth rate in 1, simply call it $g$ here.)

5. Now assume that $\varphi = 1$ and $n = 0$. Solve for the balanced growth rate of $Y$ as a function of $s, L$, and parameters. Explain why a larger $L$ implies faster balanced growth. Hint: Find expressions for ratios that are constant on the balanced growth path $(Y/K, Y/A, g_Y)$.

## 4.1 Answer: Learning by doing

**1. Balanced growth rates.** First, write all model equations in growth rates:

$$g_Y = \alpha g_K + (1-\alpha)(n + g_A) \tag{21}$$
$$g_K = sY/K \tag{22}$$
$$g_A = BY^{\varphi}/A \tag{23}$$

From (22) we find that $g_Y = g_K$ on the BGP. Then from the production function $g_Y = n + g_A$. Therefore:

$$g_Y = g_K = g_A + n = \frac{n}{1-\varphi} \tag{24}$$
$$g_A = \frac{\varphi n}{1-\varphi} \tag{25}$$

**2.** $s$ does not affect long-run growth. A one-time increase in $s$ raises $K$ and therefore $Y$. $g_A$ rises temporarily, but eventually diminishing returns in the production of ideas cause growth to slow down again.

**3.** $n$ does affect long-run growth. The reason is the same as in the R&D model. Faster output growth implies faster growth in inputs to innovation and hence faster growth of ideas.

**4. Balanced growth output** The strategy for finding levels is: Start from the production function. We need to substitute out the endogenous $K$ and $A$ to express $Y$ as a function of exogenous objects only. Use the laws of motions to express $K$ and $A$ as functions of $Y$ and other exogenous objects. Substitute these into the production function. Solve for $Y$.

The BGP capital-output ratio is

$$g_K = sY/K. \tag{26}$$

From this we find

$$K = sY/g_K. \tag{27}$$

From the law of motion for $A$:

$$A = B\ Y^{\varphi}/g_A. \tag{28}$$

Substitute into the production function:

$$Y = \left(\frac{sY}{g_Y}\right)^{\alpha} L^{1-\alpha} \left(\frac{B\ Y^{\varphi}}{g_A}\right)^{1-\alpha} \tag{29}$$

or

$$Y^{(1-\alpha)(1-\varphi)} = \left(\frac{s}{g_Y}\right)^{\alpha} L^{1-\alpha} \left(\frac{B}{g_A}\right)^{1-\alpha} \tag{30}$$

$$= s^{\alpha}\ L^{1-\alpha}\ (B/\varphi)^{1-\alpha}\ \frac{1-\varphi}{n} \tag{31}$$

**5. Balanced growth rate.** Now the balanced growth rates satisfy:

$$g = g_Y = g_K = g_A = BY/A = sY/K \tag{32}$$

Start from the law of motion for $A$:

$$\begin{aligned} g &= BY/A \\ &= B\ (K/A)^{\alpha}\ L^{1-\alpha} \end{aligned}$$

We need to find $K/A$, then we are done. From

$$BY/A = g = sY/K$$

we have

$$K/A = s/B$$

and therefore

$$g = (BL)^{1-\alpha} \; s^{\alpha} \tag{33}$$

Larger economies (higher $L$) grow faster. This is the usual scale effect. Larger economies devote more resources to innovation.

# 5 Endogenous growth without scale effects

There is one final good which is produced by combining intermediates according to the production function

$$C = \left( \int_0^B Y_i^{1/\theta} di \right)^{\theta} \tag{34}$$

We call final output $C$ because it can only be consumed. $\theta > 0$.

There are $B$ intermediate goods ($Y_i$). Each is produced from labor $L_{Yi}$ and ideas $M_i$ according to

$$Y_i = M_i^{\sigma} L_{Y,i} \tag{35}$$

Innovation must be undertaken for each variety separately:

$$\dot{M}_i = \delta M_i L_{M,i} \tag{36}$$

Finally, for reasons that we will see below, assume that $B = L^{\beta}$.

The labor constraint is now

$$L = \int_0^B (L_{Y,i} + L_{M,i}) di \tag{37}$$

We shall again assume that the labor allocation is constant such that fraction $s$ of the total labor endowment is allocated to innovation

$$sL = \int_0^B L_{Mi} di \tag{38}$$

Assume symmetry: all varieties are identical.

1. Show that
$$Y_i = M^\sigma (1-s) L^{1-\beta} \tag{39}$$
for all varieties $i$. Let's call this output level $Y$.

2. Derive an expression for the growth rate of per capita output $c = C/L$.

3. Discuss the effects of policies which change $s$ on long-run growth. How does your answer depend on the value of $\beta$. Can you provide intuition for your finding?

Note: If you have trouble with the integrals, simply replace them with sums. The result is the same.

## 5.1 Answer: Endogenous growth without scale effects

**1.** With symmetry the model equations become

$$\begin{aligned} M_i &= M \\ L_{M,i} &= sL/B = sL^{1-\beta} \\ L_{Y,i} &= (1-s)L/B = (1-s)L^{1-\beta} \end{aligned}$$

Substitute into the production function for varieties:

$$Y_i = Y = M^\sigma L_{Y,i} = M^\sigma (1-s) L^{1-\beta} \tag{40}$$

**2. Growth rate:** With symmetry:

$$\begin{aligned} C &= B^\theta Y \\ c &= C/L = B^\theta M^\sigma (1-s) L^{-\beta} \end{aligned} \tag{41}$$

Taking growth rates

$$\begin{aligned} g(c) &= \theta g(B) + \sigma g(M) - \beta n \\ g(M) &= \delta s L^{1-\beta} \\ g(B) &= \beta n \end{aligned}$$

Thus
$$g(c) = (\theta - 1)\beta n + \sigma \delta s L^{1-\beta} \tag{42}$$

**3. Long-run effects of policies** If $\beta > 1$ the model behaves much like the standard Romer model in that policies (changing $s$) do not affect long-run growth. The reason is that $L^{1-\beta} \to 0$ over time. However, if $\beta = 1$, the $L^{1-\beta}$ term does not go to 0 as $L$ becomes large. Then changes in $s$ affect long-run growth. If $\beta < 1$, then the model only has a balanced growth path for the case of $n = 0$. Otherwise it has exploding growth.

What is the intuition for the absence of scale effects? The scale effect usually arises because a larger market makes R&D more productive or more profitable. In this simple model, profitability is not an issue because $s$ is exogenous. Here, the scale effect comes from having more researchers working on each variety if there is a lot of labor around. That is

$$L_{M,i} = sL/B = sL^{1-\beta}$$

is high unless $\beta = 1$. But if $\beta = 1$, the number of varieties rises exactly in proportion to $L$, so that $L_{Mi}$ remains unchanged as $L$ grows.
How can one motivate the assumption that $B$ is proportional to $L$. Young (1998) shows that this can arise as an equilibrium outcome, if households have a preference for variety. With a larger population, per capita output rises. Consumers could increase how much they consume of each variety. Then the scale effect would be back. But with the right preferences they will hold $Y_i$ constant and increase the number of varieties instead – the scale effect is gone. However, in setting $\beta = 1$ another knife-edge assumption is introduced.

# 6 R&D: Durable Intermediates

Based on Barro & Sala-i-Martin (JPE 1992). Consider the following version of an R&D growth model where the intermediate inputs ($x$) are *durable*.

Time is continuous and goes on forever. There is a representative household who lives forever with preferences

$$\int_0^\infty e^{-\rho t} \frac{c_t^{1-\sigma} - 1}{1-\sigma} dt \tag{43}$$

The household supplies one unit of labor inelastically. The mass of households is constant and equal to one. *Final goods* are produced by competitive firms with production function

$$y_t = AL_t^{\alpha} \int_0^{N_t} x_{j,t}^{1-\alpha} dj$$

Output is used for consumption, for investment in intermediates, and for R&D. *Intermediates* are produced by monopolists. Once a new variety $j$ has been invented, the firm is endowed with $x_0$ units of $x_j$. Additional units are then accumulated according to

$$\dot{x}_{j,t} = \eta I_{j,t}^{\varphi} - \delta x_{j,t} \tag{44}$$

where for now $0 < \varphi < 1$ and $I_{j,t}$ is investment (in the form of goods) in accumulating intermediates. Intermediates depreciate at rate $\delta$. They are *rented* to final goods firms at price $R_{j,t}$.

New varieties are invented according to:

$$\dot{N} = \beta^{-1}\, z \tag{45}$$

where $z$ denotes goods devoted to R&D. There is free entry into the market for new varieties.

The solution to the household problem is standard. The budget constraint is

$$\dot{a}_t = r_t a_t - c_t \tag{46}$$

where $a$ denotes asset holdings. $c_t$ and $a_t$ solve the Euler equation

$$\dot{c}_t / c_t = \frac{r_t - \rho}{\sigma} \tag{47}$$

the budget constraint, $a_0$ given, and the TVC $\lim_{t\to\infty} e^{-\rho t} u'(c_t)\, a_t = 0$.

**Questions:**

1. Write down the problem of the final goods firm. Derive the first order conditions. Define a solution.

2. Write down the problem of an intermediate goods firm who has just invented good $j$. Derive the first-order conditions. Define a solution. Do not yet substitute out the co-state from the first-order conditions.

3. Define an equilibrium. Do not assume symmetry (there is no symmetric equilibrium because recently invented goods are supplied in smaller quantities than old ones).

4. From hereon assume $\varphi = 1$ and consider the balanced growth path with $r$ constant. Although this is not strictly speaking correct, assume that the equilibrium conditions derived for $\varphi < 1$ continue to hold (it yields the right answer). Solve for the intermediate goods firm's optimal $R(x_t)$ and $x_t$ as functions of $r$. How do they change over time?

5. Solve for the *symmetric* equilibrium values of $x_t$ and $r_t$. Given the Euler equation, you have found the equilibrium growth rate. Note: With $\varphi = 1$ there is a symmetric equilibrium because it does not take time to build up the stock of $x_j$.

#### 6.0.1 Answer: R&D: Durable Intermediates

**1. Final goods firm:** $\{y_t, L_t, x_{j,t}\}$ solve the production function and the FOCs

$$w_t = \alpha y_t / L_t \tag{48}$$

$$R_{j,t} = (1-\alpha) A L^{\alpha} x_j^{-\alpha} \tag{49}$$

**2. Intermediate goods firm:** This is really the same as the problem of a firm that owns the capital stock in the standard growth model. Period profits are $R(x)x - I$. The only difference is that the firm does not take $R$ as given - it depends on $x$.

$$V = \max \int e^{-rt} [R(x_t) x_t - I_t] dt$$

subject to

$$\dot{x} = \eta I_t^{\varphi} - \delta x \tag{50}$$

Hamiltonian:

$$H = R(x) x - I + \mu [\eta I^{\varphi} - \delta x] \tag{51}$$

FOCs:

$$\begin{aligned}\partial H/\partial I &= -1+\mu\eta\varphi I^{\varphi-1}=0\\ \dot{\mu} &= (r-\delta)\,\mu-R'(x)\,x-R(x)\end{aligned}$$

Solution: $\{I_t, x_t, \mu_t\}$ that solve 2 FOCs and law of motion for $x$. Boundary conditions: $x(0)=0$ given, $\lim_{t\to\infty} e^{-rt}\mu_t x_t = 0$.

**3. Equilibrium:** $\{R_{j,t}, x_{j,t}, N_t, I_{j,t}, \mu_{j,t}, y_t, L_t, r_t, c_t, w_t\}$ that solve:

- Household: 2
- Final goods: 3
- Intermediate goods: 3
- Free entry: Spend $\beta\ dt$ to obtain $dN=\beta/\beta\ dt$ new patents worth $V\ dt$. Equate cost and profits:

$$\beta=\int e^{-rt}\left[R(x_t)\,x_t-I_t\right]dt \tag{52}$$

- Goods market clearing: $y=c+NI+\dot{N}\beta$.
- Labor market clearing: $L=1$.
- Asset market clearing: This depends, as usual, on the structure of asset markets. If households own the firms $a_t=\int_0^{N_t} V_{j,t}$. Alternatively, one could assume that innovators issue bonds.

**4. Case $\varphi=1$:** The FOCs imply

$$\begin{aligned}\mu &= (\eta\varphi)^{-1}\\ (r+\delta)\,\mu &= R(x)\,(1-\alpha)\end{aligned}$$

Therefore, $x$ and $\mu$ must be constant over time. With a linear technology, the best approach is to build all $x$ in one shot, then keep $x$ constant.

**5. Equilibrium values:**

$$R = \frac{r+\delta}{\eta\left(1-\alpha\right)} = A\left(1-\alpha\right)L^{\alpha}x^{-\alpha}$$

Zero profit solves for $x$:

$$\begin{aligned}\beta &= x\left[R/\left(r+\delta\right)-1/\eta\right] \\ &= \frac{x}{\eta\left(1-\alpha\right)}\end{aligned}$$

Substitute into the first-order condition for $x$ to solve for $r$:

$$\left(r+\delta\right)^{1/\alpha} = \frac{L\left[\eta A\left(1-\alpha\right)^{2}\right]^{1/\alpha}}{\beta\eta\left(1-\alpha\right)} \tag{53}$$