Midterm Exam. Econ720. Fall 2017

Professor Lutz Hendricks. UNC.

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 1:15 hours.
- The total number of points is 100.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 OLG with government debt

Demographics: In each period, a unit mass of households are born. Each lives for 2 periods.

Preferences: $\ln(c_t^y) + \ln(c_{t+1}^o)$.

Endowments: the initial old hold capital k_0 and bonds b_0 (per old person). Each young works one unit of time.

Technology: Output is produced from capital k and labor l according to the aggregate production function $y_t = Ak_t + l_t$. The resource constraint is

$$c_t^y + c_t^o + k_{t+1} + g_t = Ak_t + l_t \tag{1}$$

where g_t is government consumption.

Government: The government taxes labor income at rate τ_t, issues bonds b_t, and uses the revenues to pay for government consumption and interest on outstanding debt. The budget constraint is

$$b_t + \tau_t w_t l_t = g_t + R_{t-1} b_{t-1} \tag{2}$$

where R is the gross interest rate.

Markets: goods (numeraire); labor (w); bonds (price 1); capital rental. A representative firm operates the technology.

Questions:

1. [12 points] Derive the household's consumption and savings functions.
2. [13 points] Define a competitive equilibrium. Take g_t and τ_t as given. Government debt adjusts to balance the budget.
3. [12 points] Solve for the steady state allocation.

2 Ben-Porath model

We study a decision problem in discrete time. An agent lives forever and maximizes the present value of lifetime earnings, $\sum_{t=0}^{\infty} \frac{wh_t(1-l_t)}{R^t}$, subject to

$$h_{t+1} = (1-\delta)\, h_t + F(h_t, l_t) = (1-\delta)\, h_t + (h_t l_t)^{\alpha} \tag{3}$$

The initial condition is h_0 given. Here, h is human capital, l is study time, w is a wage rate, R is a gross interest rate, and $0 < \alpha, \delta < 1$ are parameters.

Questions:

1. [12 points] Set up the Bellman equation. Derive first-order conditions and envelope conditions.
2. [9 points] Derive
$$V'(h) = w(1-l) + R^{-1} V'(h')(1-\delta) + wl \tag{4}$$
3. [9 points] Derive and interpret
$$V'(h) = w \sum_{j=0}^{\infty} \left(\frac{1-\delta}{R} \right)^{j} \tag{5}$$

3 Continuous time Ben-Porath model

Now consider the same problem as above in continuous time. The household solves

$$\max \int_0^{\infty} e^{-rt} w h_t (1-l_t)\, dt \tag{6}$$

subject to $\dot{h}_t = F(h_t, l_t) - \delta h_t$. Assume again that $F(h, l) = (hl)^{\alpha}$.

1. [12 points] Write down the Hamiltonian and derive the first-order conditions.
2. [9 points] Derive
$$\dot{\mu} = (r+\delta)\, \mu - w \tag{7}$$
3. [12 points] Show that human capital investment, hl, declines over time when hl is below the steady state.

End of exam.

4 Answer: OLG with government debt[1]

1. Household solves max utility subject to $s_{t+1} + c_t^y = w_t(1-\tau_t)$ and $c_{t+1}^o = R_{t+1}s_{t+1}$. Euler: $c_{t+1}^0 / c_t^y = R_{t+1}$. Lifetime budget constraint

$$w_t(1-\tau_t) = c_t^y + c_{t+1}^o / R_{t+1} \tag{8}$$

implies $c_t^y = c_{t+1}^o / A = w_t(1-\tau_t)/2$. The household spends half of his lifetime wealth on each good. Savings function $s_{t+1} = w_t(1-\tau_t)/2$.

2. CE: $\{c_t^y, c_t^o, s_t, k_t, b_t, \tau_t, w_t, R_{t+1}\}$ that satisfy:
   (a) household: 3
   (b) firm: $w_t = 1$, $R_t = A$
   (c) government: budget constraint
   (d) identity: $s_t = k_t + b_t$
   (e) market clearing: bonds and capital rental are implicit; $l_t = 1$; goods (RC).

3. Closed form: RC with household decision rules implies

$$(1+A)\frac{1-\tau_t}{2} = k_t(A-1) + l - g_t \tag{9}$$

The steady state government budget constraint implies

$$b = \frac{\tau - g}{A-1} \tag{10}$$

5 Answer: Ben-Porath

1. $V(h) = wh(1-l) + R^{-1}V((1-\delta)h + F(h,l))$.
   FOC: $wh = R^{-1}V(.')\,\alpha F/l$.
   Envelope: $V' = w(1-l) + R^{-1}V'(h')[1-\delta+\alpha F/h]$.

2. Substitute FOC into envelope to get the answer.

3. This is forward iteration on (4). Interpretation: An extra unit of human capital produces the discounted present value of wags given by the RHS of (4).

[1] Based on UCLA qualifying exam 2012.

6 Answer: Continuous time Ben-Porath model

1. Hamiltonian: $H = wh(1-l) + \mu[F(h,l) - \delta h]$.

   FOCs:

$$wh = \mu\alpha F/l \tag{11}$$

$$\dot{\mu} = r\mu - w(1-l) - \mu[\alpha F/h - \delta] \tag{12}$$

   The second FOC implies

$$\mu = \frac{w}{\alpha}(hl)^{1-\alpha} \tag{13}$$

   TVC: $\lim_{t\to\infty} e^{-rt}\mu_t h_t = 0$.

2. Combine the FOCs to get (7).

3. From (13), $g(\mu) = (1-\alpha)\,g(hl) = r + \delta - w/\mu$. Therefore

$$g\left([hl]^{1-\alpha}\right) = r + \delta - \alpha(hl)^{1-\alpha} \tag{14}$$

   In steady state, $g(hl) = 0$. $g(hl)$ declines in hl.

End of file.