Final Exam. Econ720. Fall 2011

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- Clearly number your answers.
- The total time is 2 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for c." Then comes the math...

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1 Growth Model With Human Capital

Demographics: A single infinitely lived household.

Preferences: $\int_0^\infty e^{-\rho t} u(c_t) dt$ with $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$.

Endowments: k, h at $t = 0$.

Technologies:

- Sector 1 produces consumption and capital goods: $G(k_1, h_1) = c + I_1$ where $I_1 = \dot{k} + \delta k$ and $k = k_1 + k_2$.
- Sector 2 produces human capital: $H(k_2, h_2) = I_2$ where $I_2 = \dot{h} + \delta h$ and $h = h_1 + h_2$.
- G and H are constant returns to scale.

Government: The government taxes capital and human capital income. It rebates revenues as a lump-sum transfer T. The budget constraint is given by $T = \tau_k q_k k + \tau_h q_h h$.

Market arrangements: There is a representative firm in each sector. Firms rent capital and human capital at rental prices q_k and q_h, respectively. Good 1 is the numeraire. The price of good 2 is p.

The household's budget constraint is given by

$$c + I_1 + pI_2 = (1 - \tau_k) q_k k + (1 - \tau_h) q_h h + T \tag{1}$$

The firms' first-order conditions are standard: $q_k = G_1 = pH_1$. $q_h = G_2 = pH_2$.

Questions:

1. [15 points] Derive the household's first-order conditions.
2. [10 points] Derive the household's Euler equation $g(c) = (r - \rho)/\sigma$ with

$$r = (1 - \tau_k)q_k - \delta = \frac{(1 - \tau_h)q_h}{p} + \frac{\dot{p}}{p} - \delta \tag{2}$$

.

3. [20 points] Derive 4 equations that solve for the balanced growth values of g, z_1, z_2, r where g is the balanced growth rate of (c, k, h) and $z_i = k_i/h_i$. Note that p is constant on the balanced growth path. Remember that, with constant returns to scale, marginal products are functions of z_i.
4. [10 points] For the special case where $H(k_{2,}, h_2) = Bh_2$ show that taxes on sector 1 do not affect the balanced growth rate. What is the intuition for this result?

2 McCall Model With Stochastic Wages

Consider a version of the McCall model where agents' wages change over time on the job.

Demographics: We study a single, infinitely lived household in partial equilibrium.

Preferences: $\sum_{t=0}^{\infty} \beta^t y_t$ where y_t is income.

Timing:

- Enter the period either as an unemployed worker (value V_U) or as employed worker with wage w (value $V(w)$).
- If unemployed, earn c and draw a wage offer with probability α.
- If employed, earn w and draw a new wage with probability λ.
- The wage offer and new wage w' are both drawn from the distribution $F(W) = \Pr(w' \le W)$ with support $[0, B]$.
- Choose whether to accept or reject w'.
- If accept: work at wage w' next period.
- If reject: be unemployed in the next period.

Employed and unemployed agents follow a reservation wage strategy with the same reservation wage $\bar{w}$. Hence $V_U = V(\bar{w}) = V(w)$ for $w \le \bar{w}$. You need not show that this is true.

Questions:

1. [15 points] State the Bellman equation for an unemployed worker.

   (a) Explain it in words.

   (b) Show that

   $$(1-\beta)V_U = c + \beta\alpha Q \tag{3}$$

   where

   $$Q = \int_{\bar{w}}^{B} (V(w') - V_U)dF(w') \tag{4}$$

2. [15 points] State the Bellman equation of an employed worker. For the continuation value, keep in mind that the worker gets a new offer w' with probability λ, but he can refuse $V(w')$ and instead choose V_U.

   (a) Explain it in words.

   (b) Show that

   $$(1-\beta)V(\bar{w}) = \bar{w} + \beta\lambda Q \tag{5}$$

3. [10 points] Find the reservation wage when $\alpha = \lambda$.

   (a) Explain what you find.

4. [5 points] Show that $\bar{w} > c$ when $\alpha > \lambda$.

   (a) What is the intuition?

3 Answers

3.1 Answer: Growth Model With Human Capital[1]

1. Hamiltonian:
$$H = u(c) + \lambda[I_1 - \delta k] + \mu[I_2 - \delta h] \tag{6}$$
where c is given by the b.c. FOC:
$$I_1 : u'(c) = \lambda \tag{7}$$
$$I_2 : u'(c)p = \mu \tag{8}$$
$$k : \dot{\lambda} = \rho\lambda + u'(c)(1 - \tau_k)q_k - \lambda\delta \tag{9}$$
$$h : \dot{\mu} = \rho\mu + u'(c)(1 - \tau_h)q_h - \mu\delta \tag{10}$$

2. The first Euler equation follows directly from first-order conditions using the standard argument. The second Euler equation follows from
$$-g(\mu) = \sigma g(c) - \dot{p}/p = \frac{(1 - \tau_h)q_h}{p} - \delta - \rho \tag{11}$$

3. Balanced growth path: g, z_1, z_2, r that satisfy:
$$g = \frac{r - \rho}{\sigma} \tag{12}$$
$$r = (1 - \tau_k)G_1(1, z_1) - \delta \tag{13}$$
$$r = (1 - \tau_h)H_2(1, z_2) - \delta \tag{14}$$
$$\frac{G_1}{G_2} = \frac{H_1}{H_2} \tag{15}$$

4. Now $H_2 = B$ so that $r = (1 - \tau_h)B - \delta$. The sector with the linear technology fixes the after-tax interest rate. Taxing sector 1 merely changes levels. z_1 adjusts to maintain equal after-tax interest rates in both sectors.

3.2 Answer: McCall Model[2]

1. Bellman equation for an unemployed worker:
$$V_U = c + \beta\left[\alpha \int \max\{V(w'), V_U\}\, dF(w') + (1 - \alpha)V_U\right] \tag{16}$$

[1] Based on Rebelo, S.; Stokey, NL (1995). Growth Effects of Flat-Rate Taxes, Journal of Political Economy, 103, 519-550.

[2] Based on Rogerson, R., Shimer, R., & Wright, R. (2005). Search-Theoretic models of the labor market: A survey. Journal of Economic Literature, 43(4), 959-988.

or

$$(1-\beta)V_U = c + \beta\alpha \int \max\{V(w') - V_U, 0\}\, dF(w') \tag{17}$$

(a) Get c today. With probability α get to choose between w' and c tomorrow.

(b) Break the integral into 2 pieces (below and above $\bar{w}$) to get the answer.

2. Bellman equation for a worker with wage w:

$$V(w) = w + \beta \left[\lambda \int \max\{V(w'), V_U\}\, dF(w') + (1-\lambda)V(w) \right] \tag{18}$$

or

$$(1-\beta)V(w) = w + \beta\lambda \int_{\bar{w}}^{B} V(w')dF(w') + \beta\lambda \int_{0}^{\bar{w}} V_U dF(w') - \beta\lambda V(w) \tag{19}$$

(a) Get w today. With probability λ, face the same choice as an unemployed worker with offer w'.

(b) Subtract V_U so that the first integral in 19 becomes Q and the second becomes 0. Now we have to add $\beta\lambda V_U$ back in, which cancels against $\beta\lambda V(\bar{w})$.

3. Reservation wage: With $V(\bar{w}) = V_U$ we have

$$(1-\beta)V_U = \bar{w} + \beta\lambda Q \tag{20}$$

$$= c + \beta\alpha Q \tag{21}$$

Value of unemployment:

$$(\alpha - \lambda)V_U = \frac{\beta}{1-\beta}[\alpha\bar{w} - \lambda c] \tag{22}$$

Perhaps easier:

$$\bar{w} - c = \beta(\alpha - \lambda)Q \tag{23}$$

If $\alpha = \lambda$: $\bar{w} = c$. The reason is that the continuation value does not depend on employment status.

4. If $\alpha > \lambda$: $\bar{w} > c$. Being unemployed has a search value. So the agent holds out for better wage offers.

Extension: Solving for the reservation wage: Add and subtract $V_U - V(w)$ in Q:

$$(1-\beta)V(w) = w + \beta\lambda \int_{\bar{w}}^{B} [V(w') - V_U]dF(w') + \beta\lambda[V_U - V(w)] \tag{24}$$

Now the integral is the same as in the V_U equation. difference the two equations to get

$$(1-\beta)[V(w) - V_U] = w - \bar{w} + \beta\lambda[V_U - V(w)] \tag{25}$$

Solve for $V(w) - V_U = \frac{w-\bar{w}}{1-\beta+\beta\lambda}$. Sub into the integral in the $V(w)$ equations and evaluate the integral. Done.