Final Exam. Econ720. Spring 2009

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 2:00 hours.
- A good answer should explain what you are doing. For example: "To find the consumption function, I take first order conditions, then use the budget constraint to solve for $c$." Then comes the math...

---

# 1 Yield Curve in the Lucas Fruit Tree Model

Consider a standard Lucas Fruit Tree model.

- Demographics: There is a representative consumer who lives forever.
- Preferences:
$$E \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$
$u$ has standard properties (strictly concave etc.).
- Endowments: The household is endowed with one tree that yields $y_t$ units of the consumption good in each period. $y_t$ is an i.i.d. random variable.
- Market arrangements: Households trade in competitive markets: (a) goods, (b) trees, (c) bonds of different maturities.
- Bond markets: A bond of maturity$i$ pays one unit of consumption $i$ periods from now. Its price is $p_{t,i}$. There are bonds for maturities $i = 1, ..., n$. These are discount bonds which do not pay interest.

**Questions:**

1. State the household's dynamic program. Hint: Think of the household as bringing bonds of maturity $0, ..., n-1$ into the period and as choosing bonds of the same maturity for next period. But note the important point: the bond $b_i$ brought into the period cost $p_i$ while the bond $b_i'$ costs $p_{i+1}$ to purchase (explain why this is true).
2. Derive first-order conditions and envelope equations. Derive Euler equations for each asset.

3. Determine the equilibrium price of bonds of maturity $i$, $p_{t,i}$. Hint: Use backward induction, starting from the bond that matures tomorrow.

4. Show that the standard Lucas asset pricing equation holds for bonds.

5. The bond's per period yield to maturity is $r_{t,i} = (1/p_{t,i})^{1/i} - 1$. For which value of $y_t$ is the yield curve flat in the sense that $r_{t,i} = r_t$ for all $i$? In which states is the yield curve upward sloping / downward sloping? Definition: The yield curve plots $r_{t,i}$ against $i$.

6. Explain the intuition for #5.

# 2 One period unemployment benefits

Consider the decision problem of a worker who lives forever and has preferences

$$E \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t c_t \tag{2}$$

The worker can be in three states:

1. employed, receiving wage $w$;
2. unemployed after being employed last period, receiving exogenous benefits $b$;
3. unemployed after being unemployed last period, receiving no benefits.

While unemployed, the worker received wage offers drawn from the distribution $F(w)$. Jobs end with probability $\delta$.

**Questions:**

1. State the Bellman equations that determine the values of being employed, unemployed after being employed, and unemployed after being unemployed last period. (Recall the general form of these Bellman equations: $rV$=[current payoff] + [prob of event] ×["capital gain"]).

2. Show that the reservation wage is independent of how long the worker has been unemployed. You need not show (yet) that the optimal plan has a reservation wage property.

3. Explain why the reservation wage is independent of how long the worker has been unemployed.

4. Show that the worker follows a reservation wage strategy. Hint: the first step is to solve for the value of being employed as a function of the wage and the value of being unemployed.

# 3 Answers

## 3.1 Answer: Yield Curve in the Lucas Fruit Tree Model

[Based on a question due to Steve Williamson]

**1. Household problem:** The household enters the period holding shares $s$ and bonds that mature $i = 0, ..., n-1$ periods from today. He chooses holdings of the same assets for tomorrow. The trick is that a bond that has maturity $i$ tomorrow has maturity $i+1$ today and costs $p_{t,i+1}$, not $p_{t,i}$.

$$V(s, b_0, ..., b_{n-1}; y) = \max u(c) + E\beta V(s', b'_0, ..., b'_{n-1}; y') \tag{3}$$

subject to the budget constraint

$$c + \sum_{i=0}^{n-1} p_{i+1} b'_i + ps' = (p+y)\, s + \sum_{i=0}^{n-1} p_i b_i \tag{4}$$

**2. First-order conditions:** Standard for the stocks, which yields the usual asset pricing equation. For the bond:

$$b'_i : \; u'(c) p_{i+1} = \beta E V_{b_i}(.') \tag{5}$$

Envelope:

$$V_{b_i} = u'(c) p_i \tag{6}$$

Euler:

$$u'(c) p_{i+1} = \beta E u'(c') p'_i$$

**3.** Solve this by backward induction:

$$p_0 = 1 \tag{7}$$

Sub that into the Euler equation and iterate to find

$$p_{t,i} = \beta^i E \frac{u'(c_{t+i})}{u'(c_t)} \tag{8}$$

with $c_t = y_t$.

**4.** Note that each asset has a standard pricing equation of the form

$$u'(c_t) = \beta^i E u'(c_{t+i}) (1 + r_{t,i})^i \tag{9}$$

where $r_{t,i}$ is not stochastic and $Eu'(c_{t+i}) = Eu'(y_{t+i})$ does not depend on the current state $y$.

**5.** The yield curve plots $\left[u'\left(c_t\right)/Eu'\left(c_{t+i}\right)\right]^{1/i}/\beta$ against $i$. It is flat the at $1/\beta$ for the value of $c_t$ that satisfies $u'\left(c_t\right)=Eu'\left(c_{t+i}\right)$. For lower $c_t$ the ratio in brackets is above 1 and the yield curve slopes down.

**6.** Intuition: As usual, low consumption relative to the future is associated with high yields. [More...]

## 3.2 Answer: One period unemployment benefits

[Based on a question due to Steve Williamson]

**1.** Start from the Bellman equations

$$\begin{aligned} V\left(w\right) &= w+\frac{1}{1+r}\left(\delta V_u^1+[1-\delta]V_e(w)\right) \\ V_u^1 &= b+\frac{1}{1+r}\int_0^{\bar{w}} max\left[V_e(w),V_u^0\right]dF(w) \\ V_u^0 &= 0+\frac{1}{1+r}\int_0^{\bar{w}} max\left[V_e(w),V_u^0\right]dF(w) \end{aligned}$$

Simplify to

$$rV_e\left(w\right) = w(1+r)+\delta\left(V_u^1-V_e(w)\right) \tag{10}$$

$$rV_u^1 = b(1+r)+\int_0^{\bar{w}} max\left[V_e(w),V_u^0\right]-V_u^1dF(w) \tag{11}$$

$$rV_u^0 = 0+\int_0^{\bar{w}} max\left[V_e(w),V_u^0\right]-V_u^0dF(w) \tag{12}$$

**2.** In (11) and (12) the "max" terms are the same. This means that the reservation wage, assuming there is one, must be the same.

**3.** Intuition: the two states only differ in the current payoff, which can no longer be affected by any worker choices. Looking forward, the two problems are the same: the worker will be in state 0 if unemployed.

**4.** From (10), we can solve for

$$V_e(w)=\frac{w(1+r)+\delta V_u^1}{r+\delta} \tag{13}$$

$V_e$ is strictly increasing in the wage. Applying that to (11) and (12) implies the reservation wage property.