Growth Rates and Logarithms: A Refresher

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U.S. Economic Growth

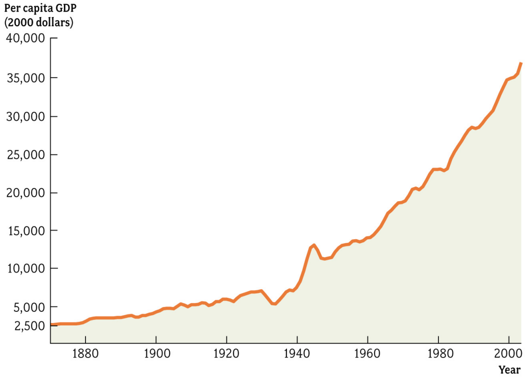


FIGURE 3.2 Per Capita GDP in the United States, 1870–2004

Macroeconomics, Charles I. Jones

It looks like U.S. growth has been accelerating? Is that true?

What Is a Growth Rate?

- We need to understand the math of growth rates.
- The growth rate g is defined as

$$g = \frac{x(t+1) - x(t)}{x(t)} \tag{1}$$

- Or:

$$x(t+1) = (1+g)x(t) \tag{2}$$

- Example: $x(t) = 100$, $g = 5\%$. Then $x(t+1) = 1.05 \times 100$.

Growth rates over multiple periods

- If we take multiple periods:

$$x(t+n) = (1+g)^n x(t) \tag{3}$$

- Example:
  - GDP per capita grows at 1.8% per year.
  - $y_{2002} = 30,000$
  - $y_{2003} = 1.018 \cdot \$30,000 = \$30,540$.
  - $y_{2003} = 1.018 y_{2002} = 1.018^2\ y_{2001}$.
- Example:
  - In 50 years, y grows by $1.018^{50} = 2.44$.

Calculating the average growth rate

- The average growth rate answers the question:
  - Which constant growth rate would change y_t to y_{t+n} in n years?
- Start from

$$y_{t+n} = y_t \cdot (1+g)^n \tag{4}$$

and solve for g.

$$(1+g)^n = y_{t+n}/y_t \tag{5}$$

$$1+g = (y_{t+n}/y_t)^{1/n} \tag{6}$$

- Example: Average GDP growth since 1870.
  - The annual growth rate is calculated from $y_{2000} = y_{1870}(1+g)^{130}$.
  - Therefore: $g = (y_{2000}/y_{1870})^{1/130} - 1 = 0.0179 = 1.79\%$ p.a.

Large long-term effects of small changes in growth

- How much lower would U.S. GDP be today, had it grown 0.5% more slowly?
- The answer:

$$\hat{y}_{2000} = y_{1870}\, 1.013^{130} = \$17,900$$

  or 46% lower than the actual 2000 level.
- **A 0.5% drop in long-run growth cuts GDP in half over 140 years.**

Logs: An easier calculation

- Average growth rate:

$$y_{t+n} = y_t \cdot (1+g)^n \tag{7}$$

- In logs:

$$\ln(y_{t+n}) = \ln(y_t) + n \ln(1+g) \tag{8}$$

- $\ln$ is the natural log: $\ln(e^x) = x$.
- For small growth rates:

$$\ln(1+g) \approx g \tag{9}$$

  (check this by example!)
- Therefore:

$$g = \frac{\ln(y_{t+n}) - \ln(y_t)}{n} \tag{10}$$

Important growth rate rules

$$g(xy) = g(x) + g(y)$$
$$g(x/y) = g(x) - g(y)$$
$$g(x^\alpha) = \alpha g(x)$$

These are easily derived from the log growth equation.

How to plot growing variables?

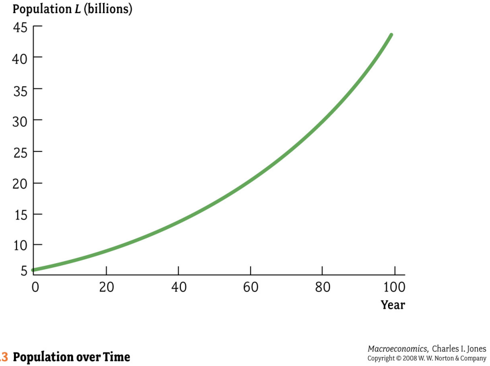


FIGURE 3.3 Population over Time

Example: Population grows at constant rate $\bar{g}$.

But the graph looks as if growth were accelerating.

Log plots

- How can we visualize that something grows at a constant rate?
  - Plot its log!
- Recall

$$\ln(y_{t+n}) = \ln(y_t) + ng \tag{11}$$

- The plot of $\ln(y_t)$ is linear with slope g.

Log plots

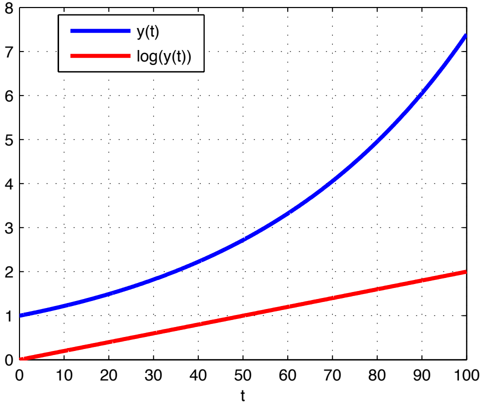
- More generally, if a variable grows at variable rate g_t:

$$y_{t+1} = y_t(1 + g_t) \tag{12}$$
$$\ln(y_{t+1}) = \ln(y_t) + g_t \tag{13}$$

- Now the plot is not linear, but the slope is still the growth rate.
- An important feature for reading log graphs: $\log(y)$ increases by 0.1 means that y roughly rises by 10%

Logarithmic scale

Example: $y(t)$ grows by 2% per year.



U.S. GDP: Log scale

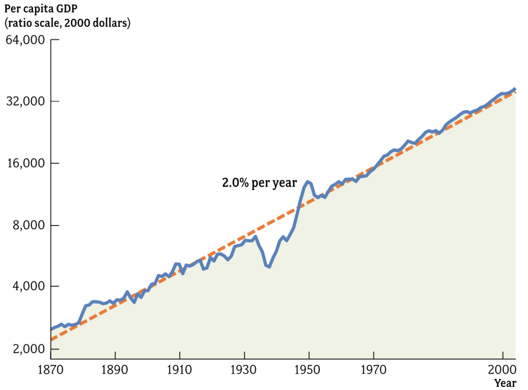


FIGURE 3.5 **Per Capita GDP in the United States, 1870–2004: Ratio Scale**

Macroeconomics, Charles I. Jones

A striking fact: Since 1870 the U.S. has grown at a constant, 2% per year rate.

Examples

Country A's GDP grows at 8% for 15 years and at -1% for 10 years. What is the average growth rate?