Practice Problems: A Model of Production

Econ520. Spring 2017. Prof. Lutz Hendricks. January 19, 2017

Jones, Macroeconomics, exercises 4.3, 4.5, 4.6.

1 Methodology

Suppose you want to find out how income taxes affect aggregate consumption. One approach would be to get data on income tax rates (τ_t) and on aggregate consumption since 1950 (C_t). Then one could run an OLS regression of the form

$$C_t = \alpha + \beta \tau_t + \varepsilon_t \tag{1}$$

1. Intuitively, what does an OLS regression do?
2. What is the interpretation of β?
3. Why does β not answer the question: a 10% increase in taxes would reduce consumption by $\beta \times 10\%$?
4. How could one answer the question: how do taxes affect consumption?

2 Production function

1. What properties of the Cobb-Douglas production function, $Y = AK^{\alpha}L^{1-\alpha}$, lead us to believe that it is a good approximation of the data?
2. How could one estimate the important parameter α?
3. For the production function $Y = AK^{\alpha}L^{\beta}$ find the marginal products of capital and labor.
4. If $\alpha + \beta = 1$, what share of income goes to capital and labor? The rest goes to pure profits. What is the profit share? Assume that capital and labor are paid their marginal products.

5. If $\alpha + \beta < 1$, what share of income goes to capital, labor, and profits?

6. If $\alpha + \beta = 1$, by how much does doubling K/L increase Y/L? By how much does a 10-fold increase of K/L increase Y/L? If $\alpha = 0.3$, why is the effect of the 10-fold increase so much less than 5 times the effect of doubling K/L?

7. Repeat the previous exercise for $\alpha = 0.8$. How does your answer change?

8. For $\alpha = 0.3$ and $\alpha = 0.8$, plot Y/L and the marginal product of capital as you vary K/L over a 10-fold range. What do you find? What does it mean for cross-country interest rate differences (keeping in mind that the real interest rate is $r = MPK - \delta$)?

2.1 Answers: Production function

1. Constant returns to scale and constant capital and labor shares.

2. Show that capital receives fraction α of total output. In the data, the share of GDP that goes to capital is about 1/3. See the slides for details.

3. See slides.

4. Capital receives α and labor receives $\beta = 1 - \alpha$. Nothing left for profits.

5. Profits get $1 - \alpha - \beta$. No change in shares that go to K and L.

6. Increase K/L by factor λ increases Y/L by factor λ^{α}. Diminishing returns to capital make added capital less and less valuable.

7. Now the production function is closer to linear. Less diminishing returns.

3 Measuring Productivity

1. Given data on capital, labor, and output, how can the production model be used to measure total factor productivity (A)?

2. Why is the value of α critical for answering the question: How important is capital for cross-country income gaps?

3.1 Answers: Measuring Productivity

1. Assume a production function. For reasons we discussed, a Cobb-Douglas function makes sense: $Y = AK^{\alpha}L^{1-\alpha}$. Get data on Y, K, L. Solve the production function for A: $A = \frac{Y}{K^{\alpha}L^{1-\alpha}}$. Plug in the data values to estimate A for each country.

2. Low α means quickly diminishing MPK. A given cross-country gap in capital implies a small gap in output. The opposite is true with high α.

4 Country comparisons

Consider two countries: the U.S. with $Y/L = \$42,000$ and $K/L = \$100,000$ and China with $Y/L = \$3,000$ and $K/L = \$6,000$. Assume the production function $Y = \bar{A}K^{1/3}L^{2/3}$.

1. The actual output gap between the U.S. and China is $42/3 = 14$. Which output gap does the model attribute to the fact that K/L in the U.S. is 16 times higher than in China?

2. How large is the ratio of $\bar{A}$ of the U.S. relative to China implied by the model?

3. Plot the production functions of the two countries (not to scale). Show the contributions of K/L and $\bar{A}$ to the Y/L gap between the 2 countries.

4.1 Answer

1. Start from the production function $y = Ak^{\alpha}$. $y_{US}/y_{CHN} = (k_{US}/k_{CHN})^{1/3} = 16^{1/3} = 2.52$. Of, if you use exact numbers: $(100/6)^{1/3} = 2.55$.

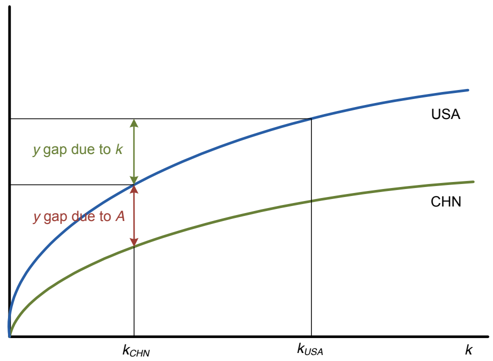

Figure 1: Decomposition of output gaps

2. Solve the production function for y and plug in numbers. Or, more easily, $14 = 2.5 \times A_{US}/A_{CHN}$ so that $A_{US}/A_{CHN} = 5.6$. Or, if you use exact numbers: $A_{US}/A_{CHN} = (42/3)/(16^{1/3}) = 5.56$.

3. See figure 1.

5 CES Production Function [Harder]

What happens to the conclusions if we relax the assumption that the production function is of the Cobb-Douglas form? Assume that

$$Y_c = A_c[\phi K_c^{\beta} + (1-\phi)L_c^{\beta}]^{1/\beta}$$

c indexes the country. Otherwise the notation is unchanged. Note that we can write

$$Y_c = A_c L_c[\phi(K_c/L_c)^{\beta} + 1 - \phi]^{1/\beta}$$

This shows that output per worker (Y/L) depends on capital per worker (K/L) and it is useful below.

This is a "CES" production function, which you should have seen in micro. The parameter β governs the elasticity of substitution between capital and labor. In case you care, that elasticity is $(1/(1-\beta))$. When $\beta \to 0$ the production function becomes Cobb-Douglas (not an obvious point, but true).

Suppose that $K_{US} = L_{US} = A_{US} = 1$, so that U.S. output is also 1. (This is just choosing units to make the math nice.)

Consider 3 values of β : 0 (Cobb-Douglas), -1 (elasticity 1/2) and 0.5 (elasticity 2).

1. Calculate the marginal products of capital and labor.
2. How much does an increase in K_c by a factor of 10 raise output? How does the answer depend on β?
3. What happens to the shares of income that go to capital and labor as you raise K? How does this depend on β? What is the intuition?

5.1 Answer

1. Marginal products: We need the chain rule for the derivative. To simplify notation, define $Q = \left[\phi K^{\beta} + (1-\phi) L^{\beta}\right]$. Then $Y = AQ^{1/\beta}$.

$$\frac{\partial Y}{\partial K} = AQ^{1/\beta - 1} \phi K^{\beta - 1} \tag{2}$$

and

$$\frac{\partial Y}{\partial L} = AQ^{1/\beta - 1} (1-\phi) L^{\beta - 1} \tag{3}$$

Usefully, the ratio of factor prices is

$$\frac{MPK}{MPL} = \frac{\phi}{1-\phi} \left(\frac{K}{L}\right)^{\beta - 1} \tag{4}$$

2. Use a calculator ...

3. It's easiest to calculate the ratio of capital income to labor income

$$\frac{MPK \times K}{MPL \times L} = \frac{\phi}{1-\phi} \left(\frac{K}{L}\right)^{\beta} \tag{5}$$

With Cobb-Douglas, $\beta = 0$ and the factor income shares are independent of K/L (as we know). If $\beta > 0$, a higher K/L increases the share of capital. With $\beta < 0$, it decreases the share of capital.

Intuition: When $\beta > 0$, the elasticity of substitution is large. Increasing K/L leads to a small decline in MPK/MPL, so the income share of K rises.

6 Comparative statics

The production model postulates the aggregate production function $Y_t = \bar{A}K_t^{\alpha}L_t^{1-\alpha}$. Assume $\alpha = 1/3$, unless stated otherwise. Countries differ in their values of the productivity parameter $\bar{A}$.

1. If $\alpha = 1/3$, how much does output per worker (Y/L) rise when K/L increases 5-fold?

2. How does your answer change when $\alpha = 2/3$? Explain the intuition underlying the difference.

3. In cross-country data, per capita GDP and capital per worker are closely related. Should one conclude that differences in capital are an important cause of differences in GDP? Explain your answer.

4. The following table shows data for 2 countries.

   (a) According to the production model, how large an output gap $\frac{Y/L_A}{Y/L_B}$ does the 20-fold capital gap $\left(\frac{K/L_A}{K/L_B}\right)$ cause?

   (b) Calculate the productivity parameters $\bar{A}$ for both countries.

Country	A	B
Y/L	100	10
K/L	400	20

6.1 Answer

1. $y = Ak^{\alpha}$. $5^{1/3} = 1.7$. This is the increase in y.

2. Now $5^{2/3} = 2.9$. The difference: MPK diminishes less quickly with the higher α.

3. No. Correlation has nothing to do with causation. k could be high because y is high or both could be high because of other common causes. This is why we need models.

4. The model attributes factor $(k_A/k_B)^{\alpha}$ to capital. $20^{1/3} = 2.7$. To calculate productivity we solve the production function for $\bar{A} = \frac{y}{k^{\alpha}}$. For A: $100/400^{1/3} = 13.6$. For B: $10/20^{1/3} = 3.7$.