Math Review Problems

Econ520 - Fall 2013 - Professor Lutz Hendricks

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All students should work out solutions to the following questions as soon as possible after the beginning of classes. The purpose is to ensure that everyone has the necessary familiarity with mathematical tools. Review any topics you have problems with.

1 Algebra

1. Simplify x^α / x^β.
2. Simplify $x^\alpha y^{-\alpha}$.
3. Simplify $e^\alpha e^\beta$.
4. Solve $ax^2 + bx + c = 0$.

1.1 Answers

1. $x^{\alpha-\beta}$.
2. $(x/y)^\alpha$.
3. $e^{\alpha+\beta}$.
4. Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2 Systems of Equations

1. Consider the following two equations:

$$\begin{aligned} a + bx + cy &= 0 \\ A + Bx + Cy &= 0 \\ a, b, c, A, B, C &> 0 \end{aligned}$$

   (a) Solve out the y variable to obtain one equation in x only.
   (b) Solve this system of equations for the solution values (x^*, y^*).
   (c) Assume that $a = 1, b = 2, c = 3$ and $A = -1, B = 2, C = -1$. Graph the solution and verify your analytical solution from (b).
   (d) Imagine that A changes to -2. Update your graph.
   (e) Imagine that B changes to 3. Update your graph.

2.1 Answer: Systems of Equations

1. Multiple equations

   (a) $[a - cA/C] + [b - cB/C]x = 0.$

   (b) Solution: $x = -\frac{a-cA/C}{b-cB/C}$.

   (c) I omit graphs.

3 Functions

1. Consider the following function:

$$\begin{aligned} a + bx + cy &= 0 \quad (1) \\ a, b, c &> 0 \quad (2) \end{aligned}$$

   (a) Graph this function. What are the values where the graph crosses the x and y axes?

   (b) Calculate the slope of the function.

   (c) Solve the function for y. That is, rewrite it in the form $y = A + Bx$. Express A and B in terms of a, b, c.

   (d) How would an increase in one of the parameters (a, b, c) change the graph of the function?

   (e) Start from some point (x^*, y^*) that satisfies the equation. If y^* increases by 2, by how much does x^* have to change? If x^* changes by 2, by how much does y^* have to change?

2. Consider the function

$$z = f(x, y)$$

   All we know about f is that it is increasing in x and y. Consider the following two points: $z_1 = f(x_1, y_1)$ and $z_2 = f(x_2, y_2)$

   (a) Suppose that $x_2 > x_1$ and $y_2 = y_1$. Can it be determined whether z_2 is greater or less than z_1?

   (b) Suppose that $x_2 > x_1$ and $y_2 < y_1$. Can it be determined whether z_2 is greater or less than z_1?

   (c) Consider all the points (x, y) for which z is some constant, let's say $z = 2$. Graph this set of points in the (x, y) plane. Is the graph upward or downward sloping?

   (d) Now imagine we raise z to 3 and again graph the combinations of (x, y) for which $f(x, y) = z = 3$. Does the new graph lie above or below the one for $z = 2$?

3. Plot $f(x) = x^\alpha$ for $0 < \alpha < 1$ and for $0 > \alpha > -1$.

4. Calculate the first and second derivatives of these functions:

   (a) $f(x) = a + bx^\alpha$.

   (b) $f(x) = a + b \ln(x)$.

(c) $f(x) = a + b \ln(c\, x)$.
(d) $f(x) = e^{bx}$.

5. Isoquants. Consider the function $f(x, y) = x^{\alpha} y^{1-\alpha}$ with $0 < \alpha < 1$. An isoquant is the set of (x, y) such that $f(x, y) = c$ where c is a constant. Plot an isoquant.

3.1 Answer: Functions

1. Properties of a function:

   (a) $x = 0 \Rightarrow y = -a/c$. $y = 0 \Rightarrow x = -a/b$.
   (b) Slope: $y = -a/c - b/c \cdot x$. Slope is $-b/c$.
   (c) See b.
   (d) $a \uparrow$: parallel shift down. $b \uparrow$: Rotate around y intercept (steeper). $c \uparrow$: Rotate around x intercept (flatter).
   (e) This is basically asking for the slope in disguise. $b \cdot \Delta x + c \cdot \Delta y = 0$. $\Delta x = 1 \Rightarrow \Delta y = -b/c$. $\Delta y = 1 \Rightarrow \Delta x = -c/b$.

2. Function of two variables.

   (a) $x_2 > x_1$ and $y_2 = y_1$: $z_2 > z_1$.
   (b) $x_2 > x_1$ and $y_2 < y_1$: indeterminate.
   (c) It is downward sloping. The slope is obtained from $f_1(x, y)\, dx + f_2(x, y)\, dy = 0$, where f_1 and f_2 are partial derivatives. Or one can simply argue: higher x raises f. Need lower y to offset this.
   (d) For any fixed x we need a higher y to raise z. The graph lies outside of the one for $z = 2$.

3. Plot omitted.

4. Derivatives.

   (a) $f'(x) = b\alpha x^{\alpha-1}$. $f''(x) = b\alpha(\alpha - 1) x^{\alpha-2}$.
   (b) $f'(x) = b/x$. $f''(x) = -b/x^2$.
   (c) $f'(x) = b/x$. $f''(x) = -b/x^2$. Because $\ln(c\, x) = \ln(c) + \ln(x)$.
   (d) $f'(x) = be^{bx}$. $f''(x) = b^2 e^{bx}$.

5. Isoquants: Solve $x^{\alpha} y^{1-\alpha} = c$ for y: $y = c^{1/(1-\alpha)} x^{-\alpha/(1-\alpha)}$. This is downward sloping, curving to the left.

4 Optimization

1. Find the maximum of $f(x) = ax - bx^2$.

2. Find the maximum of $f(x, y) = x + y^2$ subject to $x + y = 1$.

4.1 Answer: Optimization

1. Set the derivative $f'(x) = a - 2bx$ to 0 and solve for x. To make sure it's a max check that $f''(x) < 0$.

2. One approach: sub the constraint into the objective function and solve $\max_y 1 - y + y^2$.

   An alternative approach: Set up the Lagrangian

$$L = x + y^2 + \lambda(1 - x - y) \tag{3}$$

   Take the first-order conditions::

$$\begin{aligned} x : 1 - \lambda &= 0 \qquad (4) \\ y : 2y - \lambda &= 0 \qquad (5) \end{aligned}$$

   Solve for $y = x = 0.5$.

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